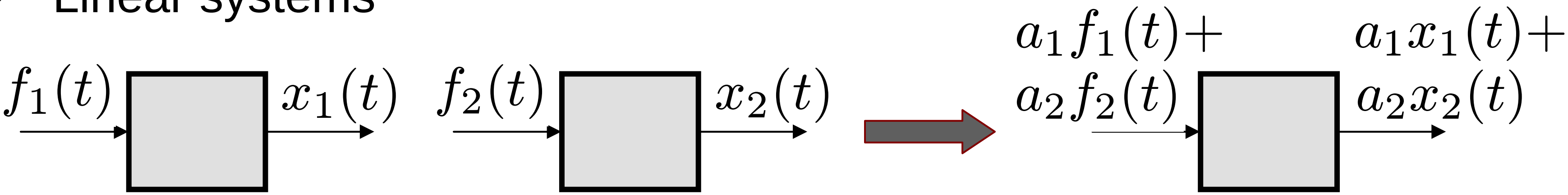
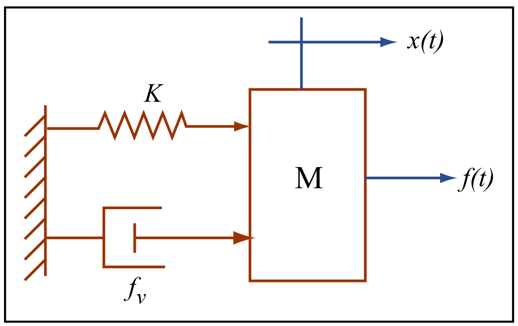
Summary from last week

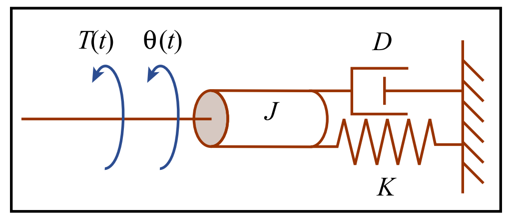
- Linear systems



- Translational & rotational mechanical elements & systems



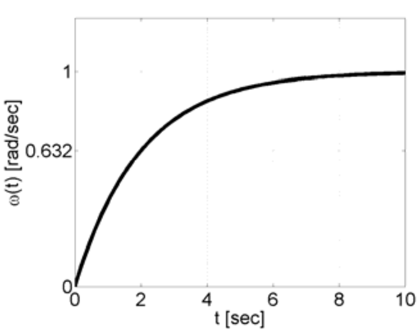
$$M\ddot{x} + f_v\dot{x} + Kx = f$$



$$J\ddot{\theta} + D\dot{\theta} + K\theta = T$$

Figures by MIT OpenCourseWare.

- Solving 1st order linear ODEs with constant coefficients



$$J\dot{\omega} + b\omega = T_0 u(t),\ \ \omega(0) = \omega_0 \qquad \Rightarrow$$

$$\omega(t) = \omega_0 \mathrm{e}^{-t/\tau} + \frac{T_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right) \qquad \omega(\infty) = \frac{T_0}{b}$$

where $\tau \equiv \dfrac{J}{b}$ time constant. steady state.

Goals for today

- Solving linear constant-coefficient ODEs using Laplace transforms
  - Definition of the Laplace transform
  - Laplace transforms of commonly used functions
  - Laplace transform properties
- Transfer functions
  - from ODE to Transfer Function
- Transfer functions of the translational & rotational mechanical elements that we know
- **Next lecture (Wednesday):**
  - Electrical elements: resistors, capacitors, inductors, amplifiers
  - Transfer functions of electrical elements
- **Lecture-after-next (Friday):**
  - DC motor (electro-mechanical element) model and its Transfer Function

Laplace transform: motivation

From ODE (linear, constant coefficients, any order) …

$$M\ddot{x}(t) + f_v\dot{x}(t) + Kx(t) = f(t)$$

input, output expressed as functions of time *t*

… to an algebraic equation

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

input, output expressed as functions of new variable *s*

Benefits:

- Simplifies solution
- *s*-domain offers additional insights
- particularly useful in control

Laplace transform: definition

Given a function $f(t)$ in the time domain we define its Laplace transform $F(s)$ as

$$F(s) = \int_{0-}^{+\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t.$$

We say that $F(s)$ is the frequency–domain representation of $f(t)$.

The frequency variable s is a complex number:

$$s = \sigma + j\omega,$$

where σ, ω are real numbers with units of frequency ($i.e.$ $\mathrm{sec}^{-1} \equiv \mathrm{Hz}$).

We will investigate the physical meaning of σ, ω later when we see examples of Laplace transforms of functions corresponding to physical systems.

Example 1: Laplace transform of the step function

Consider the step function (*aka* Heaviside function)

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

According to the Laplace transform definition,

$$\begin{aligned} U(s) &= \int_{0-}^{+\infty} u(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{0-}^{+\infty} 1 \cdot \mathrm{e}^{-st}\mathrm{d}t = \\ &= \left(\frac{1}{-s}\mathrm{e}^{-st}\right)\Bigg|_{0-}^{+\infty} = \frac{1}{-s}\Big(0 - 1\Big) = \\ &= \frac{1}{s}. \end{aligned}$$

Interlude: complex numbers: what does 1/s mean?

Recall that $s = \sigma + j\omega$. The real variables σ, ω (both in frequency units) are the real and imaginary parts, respectively, of s. (We denote $j^2 = -1$.)

Therefore, we can write

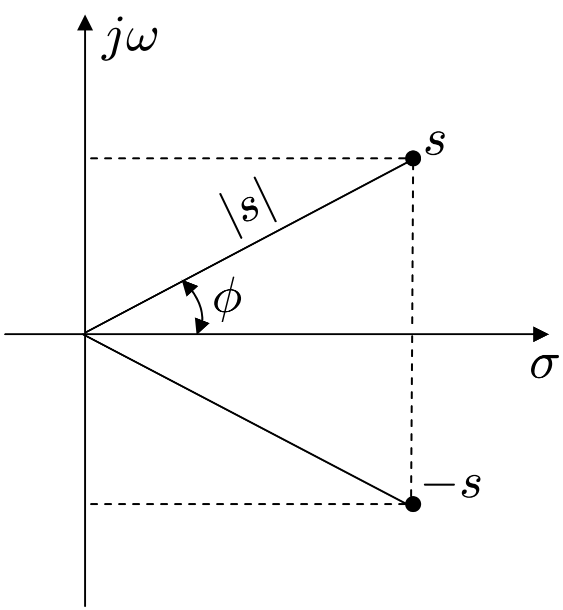
$$\frac{1}{s} = \frac{1}{\sigma + j\omega} = \frac{\sigma - j\omega}{(\sigma + j\omega)(\sigma - j\omega)} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2}.$$

Alternatively, we can represent the complex number s in polar form $s = |s|\,\mathrm{e}^{j\phi}$,

where $|s| = \left(\sigma^2 + \omega^2\right)^{1/2}$ is the magnitude and $\phi \equiv \angle s = \operatorname{atan}(\omega/\sigma)$ the phase of s.

It is straightforward to derive

$$\frac{1}{s} = \frac{1}{|s|}\mathrm{e}^{-j\phi} \Rightarrow \left|\frac{1}{s}\right| = \frac{1}{|s|} \qquad \text{and} \qquad \angle\frac{1}{s} = -\angle s.$$



Example 2: Laplace transform of the exponential

Consider the decaying exponential function beginning at $t = 0$

$$f(t) = \mathrm{e}^{-at}u(t),$$

where $a > 0$ (note the presence of the step function in the above formula.)

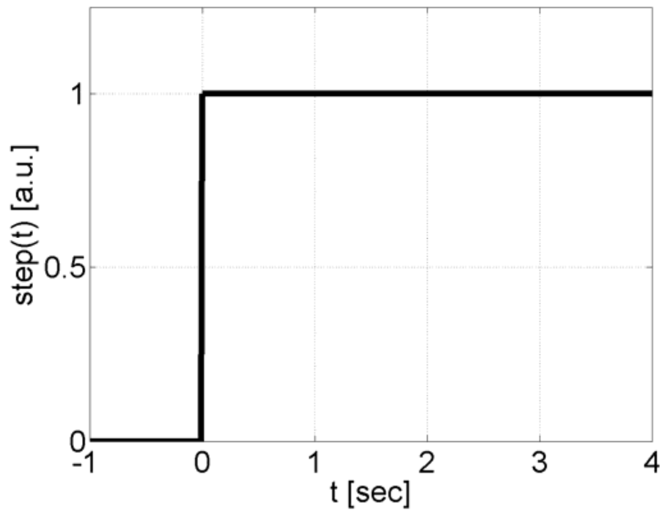
Again we apply the Laplace transform definition,

$$\begin{aligned} F(s) &= \int_{0-}^{+\infty} \mathrm{e}^{-at}u(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{0-}^{+\infty} \mathrm{e}^{-(s+a)t}\mathrm{d}t = \\ &= \left(\frac{1}{-(s+a)}\mathrm{e}^{-(s+a)t}\right)\Bigg|_{0-}^{+\infty} = \frac{1}{-(s+a)}\Big(0-1\Big) = \\ &= \frac{1}{s+a}. \end{aligned}$$

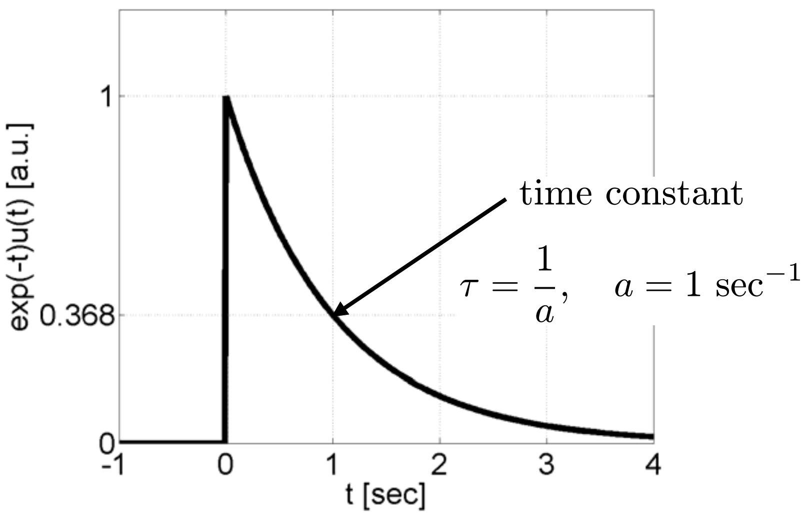
Laplace transforms of commonly used functions

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Figure by MIT OpenCourseWare.



Step function (*aka* Heaviside)



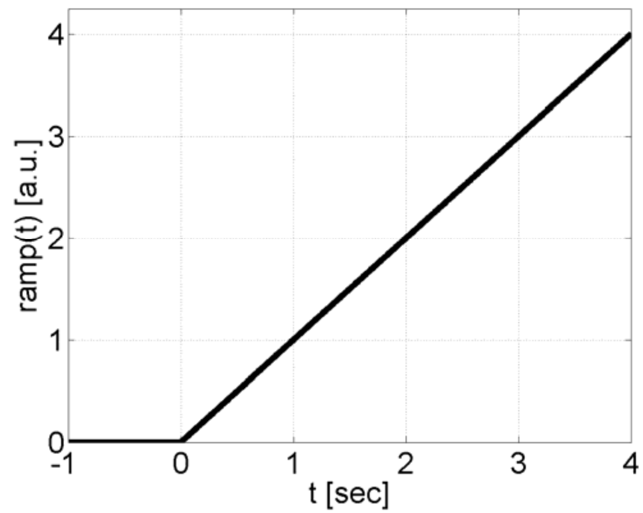
Nise Table 2.1

Laplace transforms of commonly used functions

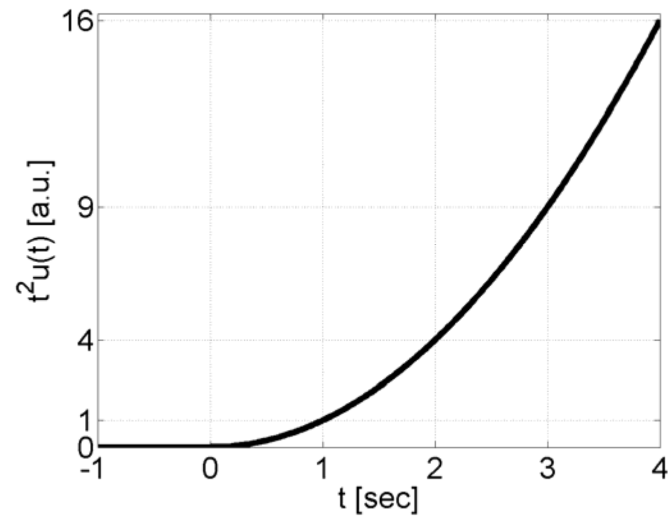
Polynomials

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^n + 1}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Figure by MIT OpenCourseWare.



Ramp function



Quadratic function

$n = 2$

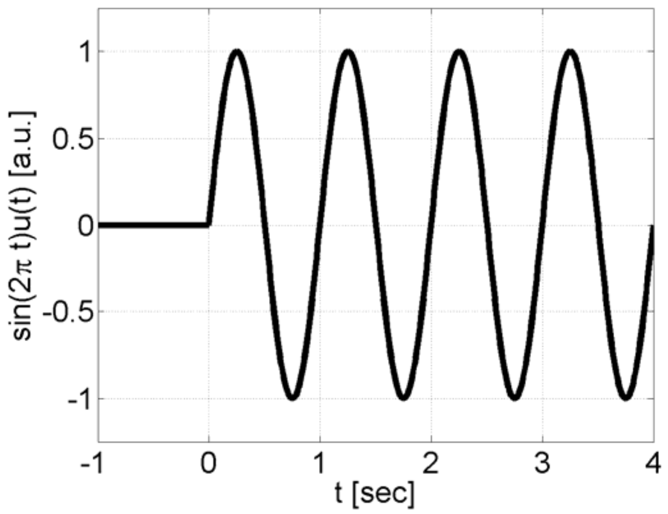
Nise Table 2.1

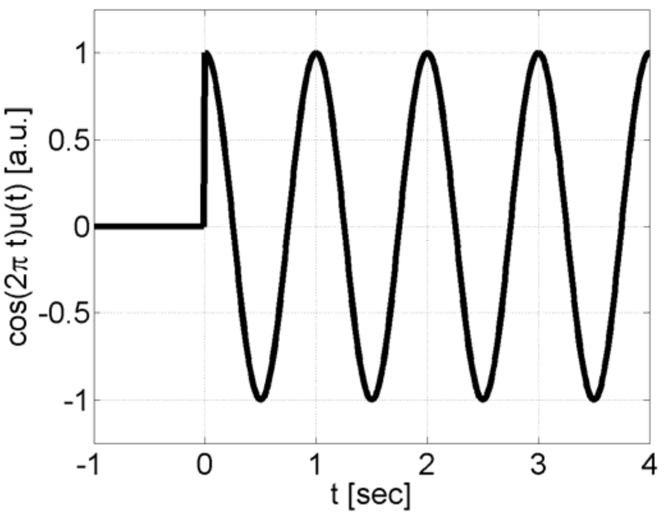
Laplace transforms of commonly used functions

Sinusoids

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Figure by MIT OpenCourseWare.





Nise Table 2.1

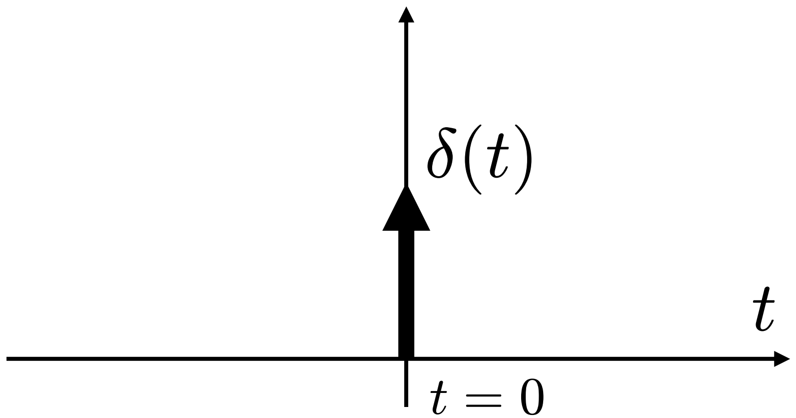
Laplace transforms of commonly used functions

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^n + 1}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Figure by MIT OpenCourseWare.

Nise Table 2.1

Impulse function (*aka* Dirac function)



It represents a pulse of

- infinitessimally small duration; and
- finite energy.

Mathematically, it is defined by the properties

$$\int_{-\infty}^{+\infty} \delta(t) = 1; \qquad \text{(unit energy) and}$$

$$\int_{-\infty}^{+\infty} \delta(t) f(t) = f(0) \qquad \text{(sifting.)}$$

Properties of the Laplace transform

Let $F(s)$, $F_1(s)$, $F_2(s)$ denote the Laplace transforms of $f(t)$, $f_1(t)$, $f_2(t)$, respectively. We denote $\mathcal{L}[f(t)] = F(s)$, etc.

- Linearity

$\mathcal{L}[K_1 f_1(t) + K_2 f_2(t)] = K_1 F_1(s) + K_2 F_2(s)$,
where K_1, K_2 are complex constants.

- Differentiation

The differentiation property is the one that we'll find most useful in solving linear ODEs with constant coeffs.

  - $\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = sF(s) - f(0-)$;
  - $\mathcal{L}\left[\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right] = s^2 F(s) - sf(0-) - \dot{f}(0)$; and
  - $\mathcal{L}\left[\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0-)$.

- Integration

$$\mathcal{L}\left[\int_{0-}^{t} f(\xi)\mathrm{d}\xi\right] = \frac{F(s)}{s}.$$

A more complete set of Laplace transform properties is in Nise Table 2.2.
We'll learn most of these properties in later lectures.

Inverting the Laplace transform

Consider

$$F(s) = \frac{2}{(s+3)(s+5)}. \tag{1}$$

We seek the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\left[F(s)\right]$:*i.e.*, a function $f(t)$ such that $\mathcal{L}\left[f(t)\right] = F(s)$.

Let us attempt to re–write $F(s)$ as

$$F(s) = \frac{2}{(s+3)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+5}. \tag{2}$$

That would be convenient because we know the inverse Laplace transform of the $1/(s+a)$ function (it's a decaying exponential) and we can also use the linearity theorem to finally find $f(t)$. All that'd be left to do would be to find the coefficients K_1, K_2.

This is done as follows: first multiply both sides of (2) by $(s+3)$. We find

$$\frac{2}{s+5} = K_1 + \frac{K_2(s+3)}{s+5} \overset{s=-3}{\Longrightarrow} K_1 = \frac{2}{-3+5} = 1.$$

Similarly, we find $K_2 = -1$.

Inverting the Laplace transform

So we have found

$$F(s) = \frac{2}{(s+3)(s+5)} = \frac{1}{s+3} - \frac{1}{s+5}.$$

From the table of Laplace transforms (Nise Table 2.1) we know that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] &= \mathrm{e}^{-3t}u(t) \qquad \text{and} \\ \mathcal{L}^{-1}\left[\frac{1}{s+5}\right] &= \mathrm{e}^{-5t}u(t). \end{aligned}$$

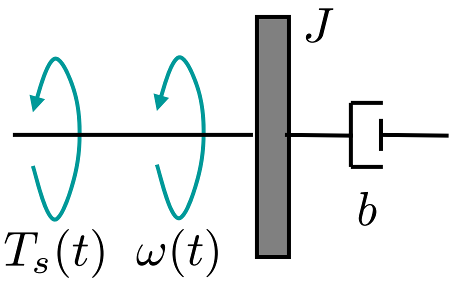
Using these and the linearity theorem we obtain

$$\mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\frac{2}{(s+3)(s+5)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+3} - \frac{1}{s+5}\right] = \mathrm{e}^{-3t} - \mathrm{e}^{-5t}.$$

The process we just followed is known as **partial fraction expansion.**

Use of the Laplace transform to solve ODEs

- Example: motor-shaft system from Lecture 2 (and labs)



$$J\dot{\omega}(t) + b\omega(t) = T_s(t),$$

$$\text{where } T_s(t) = T_0 u(t) \qquad \text{(step function)}$$

$$\text{and } \omega(t=0) = 0 \qquad \text{(no spin–down)}.$$

Taking the Laplace transform of both sides,

$$Js\Omega(s) + b\Omega(s) = \frac{T_0}{s} \Rightarrow \Omega(s) = \frac{T_0}{b}\frac{1}{s\left((J/b)s+1\right)} = \frac{T_0}{b}\frac{1}{s(\tau s+1)},$$

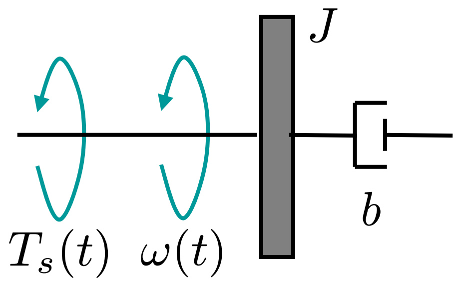
where $\tau \equiv J/b$ is the time constant (see also Lecture 2).

We can now apply the partial fraction expansion method to obtain

$$\Omega(s) = \frac{T_0}{b}\left(\frac{K_1}{s} + \frac{K_2}{\tau s+1}\right) = \frac{T_0}{b}\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right) = \frac{T_0}{b}\left(\frac{1}{s} - \frac{1}{s+(1/\tau)}\right).$$

Use of the Laplace transform to solve ODEs

- Example: motor-shaft system from Lecture 2 (and labs)



$$J\dot{\omega}(t) + b\omega(t) = T_s(t),$$

where $T_s(t) = T_0 u(t)$ (step function)

and $\omega(t=0) = 0$ (no spin–down).

We have found

$$\Omega(s) = \frac{T_0}{b}\left(\frac{1}{s} - \frac{1}{s + (1/\tau)}\right).$$

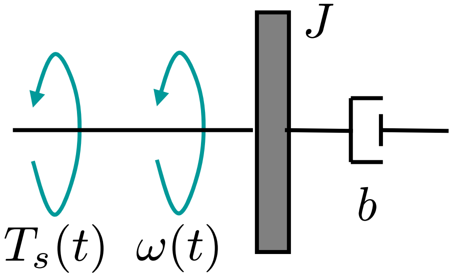
Using the linearity property and the table of Laplace transforms we obtain

$$\omega(t) = \mathcal{L}^{-1}\left[\Omega(s)\right] = \frac{T_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right),$$

in agreement with the time–domain solution of Lecture 2.

Transfer Functions

- Consider again the motor-shaft system :



$$J\dot{\omega}(t) + b\omega(t) = T_s(t),$$

where now $T_s(t)$ is an arbitrary function,

but still $\omega(t=0) = 0$ (no spin–down).

Proceeding as before, we can write

$$\Omega(s) = \frac{T_s(s)}{Js+b} \Leftrightarrow \frac{\Omega(s)}{T_s(s)} = \frac{1}{Js+b}.$$

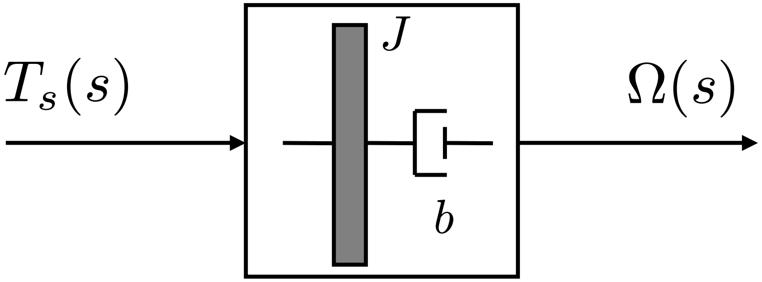
Generally, we define the ratio

$$\frac{\mathcal{L}\left[\text{output}\right]}{\mathcal{L}\left[\text{input}\right]} = \textbf{Transfer Function}\text{; in this case, } \mathrm{TF}(s) = \frac{1}{Js+b}.$$

We refer to the $(\mathrm{TF})^{-1}$ of a single element as the **Impedance** $Z(s)$.

Transfer Functions in block diagrams

$$\text{Transfer Function}(s) := \frac{C(s)}{R(s)}.$$



$$\text{TF}(s) := \frac{\Omega(s)}{T_s(s)} = \frac{1}{Js+b}.$$

Important: To be able to define the Transfer Function, the system ODE must be linear with constant coefficients.

Such systems are known as **Linear Time-Invariant**, or **Linear Autonomous**.

Impedances: rotational mechanical

Table removed due to copyright restrictions.

Please see: Table 2.5 in Nise, Norman S. *Control Systems Engineering.* 4th ed. Hoboken, NJ: John Wiley, 2004.

(In the notes, we sometimes use b or B instead of D.)

Impedances: translational mechanical

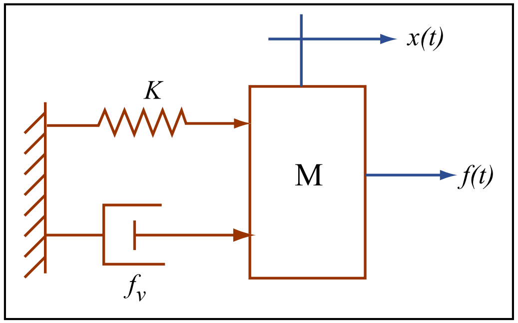
Table removed due to copyright restrictions.

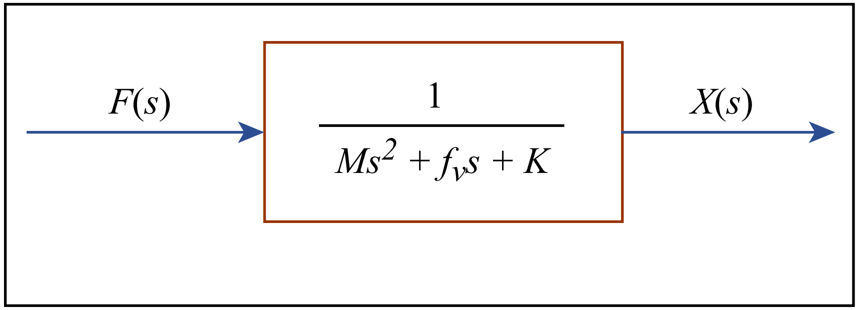
Please see Table 2.4 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

(In the notes, we sometimes use b or B instead of f_v.)

Transfer Functions: multiple impedances

System ODE: $M\ddot{x}(t) + f_v\dot{x}(t) + Kx(t) = f(t).$





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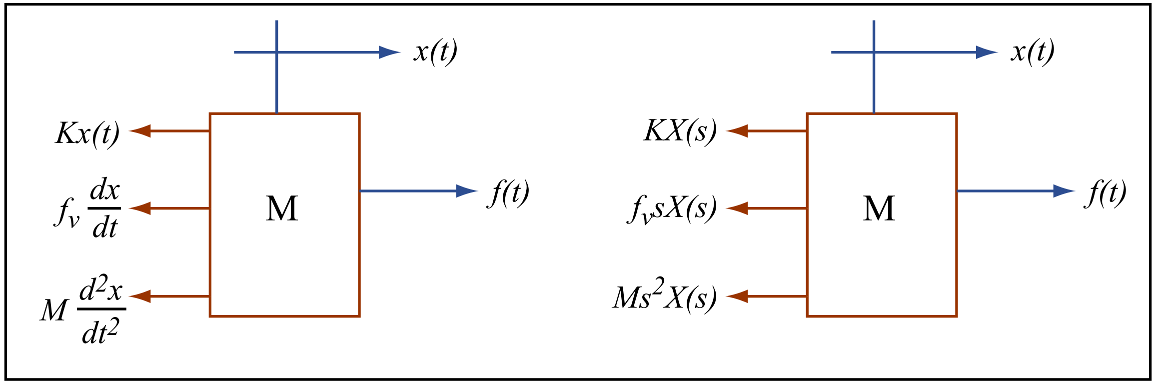
$$\left[\sum \text{Impedances}\right] X(s) = \left[\sum \text{Forces}\right].$$



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Summary

- Laplace transform

$$\mathcal{L}\left[f(t)\right] \equiv F(s) = \int_{0-}^{+\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t.$$

$$\mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0-).$$

$$\mathcal{L}\left[u(t)\right] \equiv U(s) = \frac{1}{s}.$$

$$\mathcal{L}\left[\int_{0-}^{t} f(\xi)\mathrm{d}\xi\right] = \frac{F(s)}{s}.$$

$$\mathcal{L}\left[\mathrm{e}^{-at}\right] = \frac{1}{s+a}.$$

- Transfer functions and impedances

$$J\ddot{\theta}(t) = T(t) \Rightarrow Z_J = Js^2; \quad f_v\dot{\theta}(t) = T(t) \Rightarrow Z_{f_v} = f_v s; \quad K\theta(t) = T(t) \Rightarrow Z_K = K.$$

$$J\dot{\omega}(t) + b\omega(t) = T_s(t) \stackrel{\mathcal{L}}{\Longrightarrow} (Js+b)\,\Omega(s) = T_s(s) \Rightarrow \frac{\Omega(s)}{T_s(s)} \equiv \mathrm{TF}(s) = \frac{1}{Js+b}.$$

$$M\ddot{x}(t) + f_v\dot{x}(t) + Kx(t) = f(t) \Rightarrow \frac{X(s)}{F(s)} \equiv \mathrm{TF}(s) = \frac{1}{Ms^2 + f_v s + K}.$$