Response of 1st-order system to sinusoidal input

$\tau \dot{x} + x = f(t)$
$x(0) = 0 \rightarrow$ initial condition
$f(t) = f_0 \cos \omega_0 t \rightarrow$ periodic forcing function
$\omega_0 =$ angular frequency $\left[\frac{rad}{sec}\right]$
$\nu_0 = \frac{\omega_0}{2\pi} =$ frequency [Hz] $= \left[sec^{-1}\right]$
$f_0 =$ amplitude

Solution: $x(t) = x_h(t) + x_p(t) =$ homogeneous + particular

$x_h(t) = Ae^{-\frac{t}{\tau}}$

Conjecture: $x_p(t) = \alpha f_0 \cos(\omega_0 t + \psi)$

Procedure: First calculate α, ψ, then A.

$\tau \dot{x_p}(t) + x_p(t) = f(t) \Rightarrow$
$-\tau\omega_0 \alpha f_0 \sin(\omega_0 t + \psi) + \alpha f_0 \cos(\omega_0 t + \psi) = f_0 \cos \omega_0 t \Rightarrow$ trig substitution
$-\tau\omega_0 \alpha f_0 [\sin \omega_0 t \cos \psi + \cos \omega_0 t \sin \psi] + \alpha f_0 [\cos \omega_0 t \cos \psi - \sin \omega_0 t \sin \psi] = f_0 \cos \omega_0 t \Rightarrow$
$-\alpha f_0(\omega_0 \tau \cos \psi + \sin \psi) \sin \omega_0 t + \alpha f_0(-\omega_0 \tau \sin \psi + \cos \psi) \cos \omega_0 t = f_0 \cos \omega_0 t \Rightarrow$ must be true for all t

$\Rightarrow$ equate coefficients

$$\begin{cases} \alpha f_0(-\omega_0 \tau \sin \psi + \cos \psi) = f_0 & (1) \\ \omega_0 t \cos \psi + \sin \psi = 0 & (2) \end{cases}$$

From (2), $\Rightarrow \tan \psi = -\omega_0 t$
From (1) $\Rightarrow \tan \psi \sin \psi + \cos \psi = \frac{f_0}{\alpha f_0} \Rightarrow \frac{1}{\cos \psi} = \frac{1}{\alpha}$

$\Rightarrow \alpha = \cos \psi = \frac{1}{\sqrt{1+\tan^2 \psi}} = \frac{1}{\sqrt{1+(\omega_0 t)^2}}$

$\therefore x_p(t) = \frac{f_0}{\sqrt{1+\tan^2 \psi}} \cos(\omega_0 t - \tan^{-1} \omega_0 t)$

Back to the complete solution:

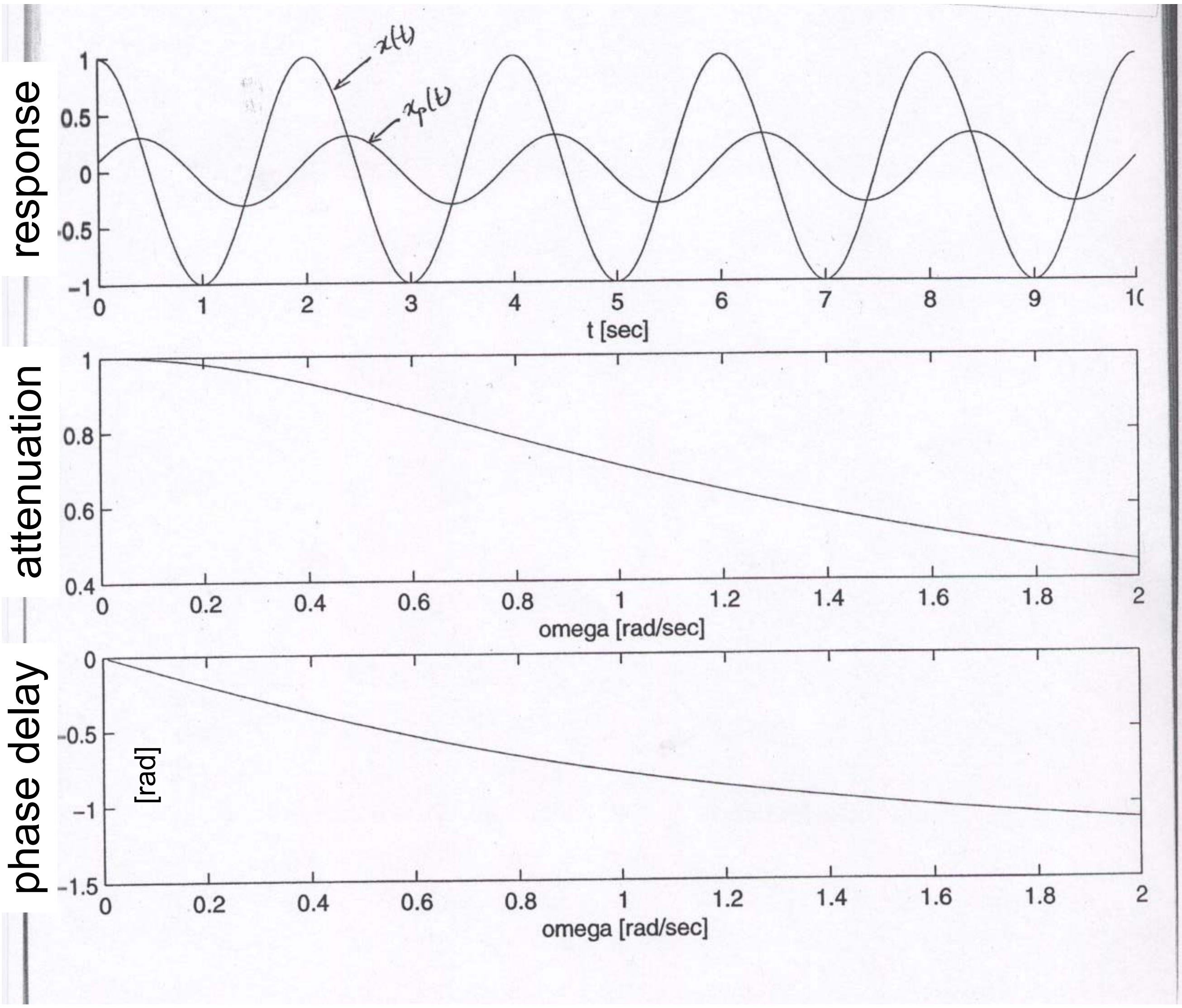
$x(t) = Ae^{-\frac{t}{\tau}} + \alpha f_0 \cos(\omega_0 t - \psi)$
$x(0) = 0 = A + \alpha f_0 \cos \psi \Rightarrow A = -\alpha f_0 \cos \psi$

Final Solution:

$$x(t) = \frac{f_0}{\sqrt{1+(\omega_0 t)^2}}\left(-\frac{e^{-\frac{t}{\tau}}}{\sqrt{1+(\omega_0 t)^2}} + \cos(\omega_0 t - \psi)\right)$$

Note that the first term here is the exponential decay, while the second is the steady-state solution. Long-term, we are interested in the steady-state response (i.e., $t \gg \tau$) when the exponential has decayed and the cosinusoidal is what remains.

$$x_{steady-state}(t) \simeq \frac{f_0}{\sqrt{1+(\omega_0 t)^2}} \cos(\omega_0 t - \psi)$$


response
x(t)
$x_p(t)$
t [sec]
attenuation
omega [rad/sec]
phase delay
[rad]
omega [rad/sec]

More generally, for linear time-invariant systems, where $f(t) \rightarrow \boxed{LTI} \rightarrow x(t)$ [steady-state only!]:

$$\text{If } f(t) = f_0 \cos(\omega_0 t - \alpha) \text{ then } x(t) = f_0 \cos(\omega_0 t - \alpha + \psi),$$

since the system is linear ($\omega_0 t$) and shift invariant (α).

E.g.: 1st-order low-pass system, $\tau \dot{x} + x = f$

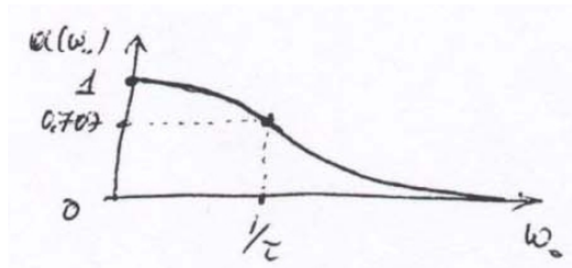


Figure 1: $\alpha(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}}$
A graph showing the change in α as ω ranges from zero to ω_0.

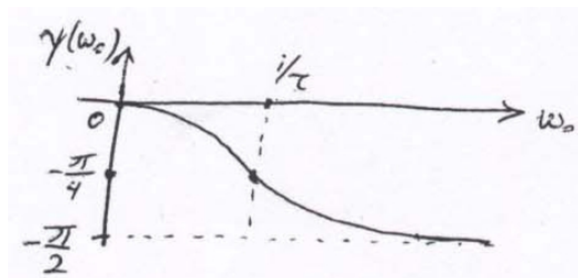


Figure 2: $\psi(\omega) = \tan^{-1}(-\omega\tau)$
A graph of ψ over the range of ω from zero to ω_0.

It is convenient here to define a complex number, G:

$$G(\omega) = \alpha(\omega) e^{i\psi(\omega)}$$

where $\alpha(\omega)$ is the magnitude of the function, and $\psi(\omega)$ is the phase. G therefore is the transfer function.

Why is it convenient?

We started this discussion by using the excitation:

$$f(t) = f_0 \cos \omega_0 t = f_0 \text{Re}[e^{i\omega t}]$$

We found that:

$$x(t) = f_0 \alpha \cos \omega_0 t + \psi = f_0 \text{Re}[\alpha e^{i(\omega t + \psi)}]$$

In other words:

$$x(t) = f_0 \text{Re}[H(\omega_0) e^{i\omega_0 t}] \text{ !!}$$

We will soon return to this point!

The representation of a sinusoid

$$\alpha \cos \omega_0 t + \psi$$

by a complex number

$$\alpha e^{i\psi}$$

is known as **phasor representation**.
Sometimes we use the notation

$$\alpha e^{i\psi} \equiv \alpha \angle \psi$$

to denote the phasor in terms of its **amplitude** α and **phase** ψ.