

2.004 *Dynamics and Control II*

**Laboratory Session 5:**  
**Elimination of Steady-State Error Using Integral Control Action**<sup>1</sup>

**Laboratory Objectives:**

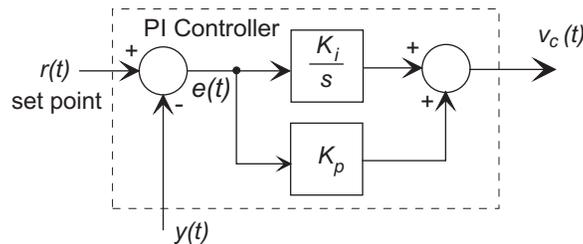
- (i) To investigate the elimination of steady-state error through the use of *integral* (I), and *proportional plus integral* (PI) control.
- (ii) To compare your experimental results with a Simulink digital simulation.

**Introduction:** In the previous laboratory experiments you have noted that there was a steady-state error to a constant angular velocity command, and that the error magnitude depended on the degree of viscous damping present. In many control problems it is desirable to eliminate the steady-state error, and the most common way of doing this is through the use of *integral* control action and *proportional plus integral* (PI) control.

A PI controller has a transfer function

$$G_c(s) = K_p + K_i \frac{1}{s}$$

with a block-diagram



and a time domain response

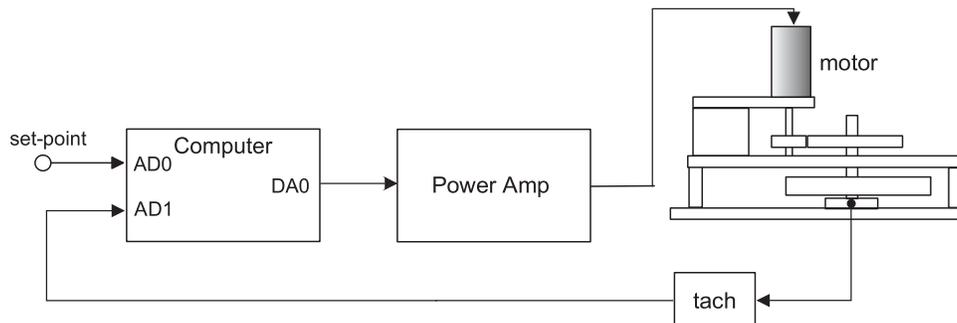
$$v_c(t) = K_p e(t) + K_i \int_0^t e(t) dt$$

where  $v_c(t)$  is the controller output. A description of how the integral component acts to eliminate steady-state error is given in Appendix A. Please take a few minutes to read through and understand the Appendix.

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<sup>1</sup>October 22, 2007

**The Experimental Setup:** The set-up is the same as in Lab. 4, using the 2.004 PID Controller as shown below:



Use the tachometer low-pass filter as you did in Lab 4. In this lab, in addition to proportional control you will be using integral control, adjusted by the knob labelled **Int. Gain (Ki)** on the front panel. In digital control systems such as this, real-time integration is done through an approximate numerical algorithm, such as rectangular integration, where the integral is represented as a sum  $s_n$

$$s_n = s_{n-1} + e_n \Delta T$$

where  $e_n$  is the error at the  $n$ th iteration, and  $\Delta T$  is the time step, or trapezoidal integration

$$s_n = s_{n-1} + (e_{n-1} + e_n) \Delta T / 2$$

**Experiment #1: Verification of Integrator Performance** Verify that the integrator is functioning correctly using the following steps:

- Connect the computer-based controller, but keep the power amp turned off for all parts of this experiment.
- Set the function generator to produce a step (square) function of amplitude 1 volt, at a frequency of 1 Hz.
- Open the controller, and select a sampling rate of 100 samples/sec. (Maintain this value for all parts of the lab.)
- Set  $K_p = 0$  and  $K_i = 1$  on the front panel. Start the controller and observe the error trace. (If the tachometer is noisy, you might want to disconnect it and ground the input). Visually confirm that the error trace is the integral of the input. Either save and plot the output, or make a sketch of it.
- Add a 0.5 volt offset to the square wave and repeat part (d).
- Now set a 1 Hz. triangular wave (no offset) on to the function generator and repeat the experiment.

**Experiment #2: Proportional Control** Obtain a “baseline” step response with proportional control. Basically repeat the Lab. 4 step response measurement to demonstrate the existence of the steady-state error:

- (a) Set  $K_p = 3$ , and  $K_i = 0$ , with a sampling rate of 100 samples/second. Install one damping magnet.
- (b) Set the function generator to produce a DC signal of 1 volt magnitude.
- (c) Record and plot the closed-loop step response, and measure the steady-state error.

**Experiment #3: Pure Integral Control**

- (a) Now investigate pure integral control by repeating Expt. #2 with  $K_p = 0$ , and  $K_i = 3$  so that

$$G_c(s) = \frac{3}{s}.$$

When using integral control, make sure that the power amp is turned on *before* starting the controller. This avoids the problem of “integrator wind-up.”

Has integral control helped with the steady-state error? Can you tell? What has happened to the transient response? Plot your results.

- (b) Remove the damping magnet and repeat part (a). Is the response “better” or “worse”.

Discuss your results with your lab instructor. Look at the closed-loop characteristic equation from Appendix A, and discuss how the closed-loop roots are affected by the values of  $B$  and  $K_i$ . In particular, think about what happens to the closed-loop if the viscous damping  $B = 0$ .

**Experiment #4: PI Control:** In this experiment, use PI Control, that is with

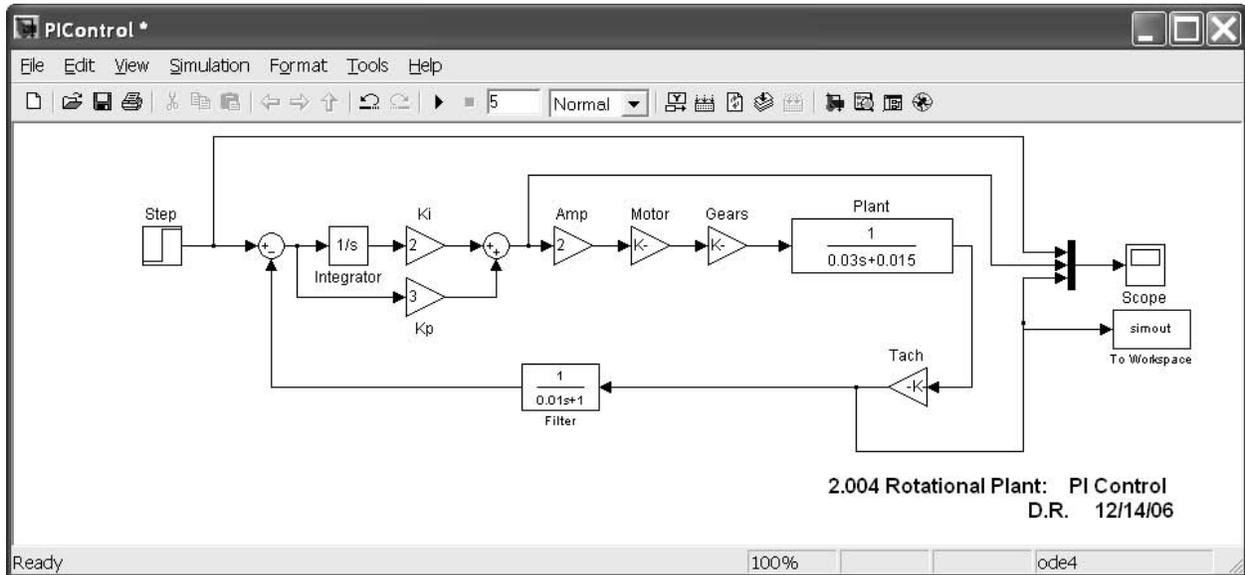
$$G_c(s) = K_p + K_i \frac{1}{s} = \frac{K_p s + K_i}{s} = K_p \left( \frac{s + K_i/K_p}{s} \right)$$

- (a) Start with  $K_p = 3$ ,  $K_i = 1$ , and a single magnet for damping. Use the same function generator settings, and record and save the step response. (Note – use the pan and zoom tools to select a complete positive step section of the response before saving it to MATLAB.) Is the response more satisfactory?
- (b) Repeat (a) with  $K_i = 5$  and 10. In each case save the response to MATLAB, and make a plot of the positive step response.
- (c) Qualitatively examine the effect of integral control by using a finger to add a constant disturbance torque to the flywheel. Observe the controller output (blue/grey trace). Make a note of what happens.

Compare your three plots. Briefly describe how the value of  $K_i$  has affected 1) any “overshoot” in the step response, 2) the time to the peak response, and the time to reach the steady-state response.

**Experiment #5: Compare your results with a Simulink Simulation:** Simulink is one of the most widely used computer tools for control system analysis and design. It is an integral part of MATLAB, and is a drag-and-drop block-diagram time-domain simulation

language. Simulink provides a graphical work-space where you can create very complex system models without writing a single line of code. Later in this course we will introduce you to programming in Simulink, but for now we provide you with a Simulink model of the lab setup and ask you to run it and compare your experimental step-responses with the Simulink simulation.



The figure above shows the "pre-wired" Simulink simulation for this lab. You can change the values of  $K_p$  and  $K_i$  by double-clicking on the appropriate block and entering the new value. You can display the 'scope' by double-clicking on the icon, and then resizing the window. The input block at the far left is a Simulink *step function*, so that the simulation will display the closed-loop step-response. Many other functions may be found in the "sources" library. Three signals are "multiplexed" on to the scope (input, controller output, and tach output). In addition, the tach output is connected to a block labelled "simout". This writes a vector named *simout* to the MATLAB workspace so that you may access the response in MATLAB. You can change the name of the MATLAB variable by double clicking on the icon.

To run the simulation, simply click on the right-arrow in the toolbar.

- (a) The Simulink model is contained in the file *PIControl.mdl* in the Lab 5 folder of the 2.004 Course Locker on the lab machines. To run the model, drag the file to your desktop or home directory (Z:). Double-click on the file to start MATLAB and open the model.
- (b) Run the simulation for the case of PI control with  $K_p = 3$ , and  $K_i = 1, 5, 10$ . Save the output to a different variable name in each case.
- (c) Compare your experimental and simulated data. If you can, make a single plot for each of the three conditions with the real and simulated data.

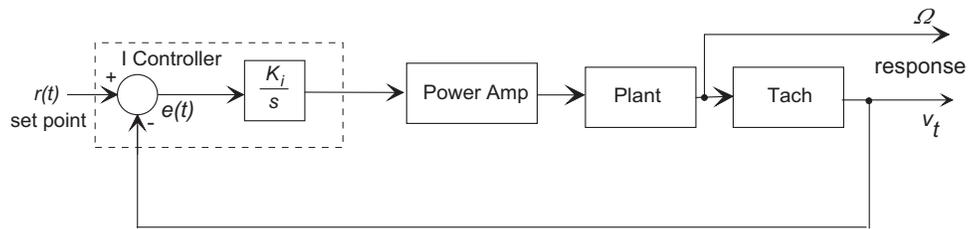
## Appendix A: Introduction to Integral Control Action

In the previous labs we have noted that there is a steady-state error in the angular velocity of the plant when there is a viscous disturbance torque present. Integral control action is very commonly used to eliminate the steady state error.

**Pure Integral Control:** Assume that we replace our proportional controller with an integrator with gain  $K_i$  so that the controller output is

$$\begin{aligned} v_c(t) &= K_i \int_0^t e(t) dt + v_c(0) \\ &= K_i \int_0^t (r(t) - y(t)) dt + v_c(0) \end{aligned}$$

where  $e(t) = r(t) - y(t)$  is the error. For simplicity also assume that  $v_c(0) = 0$ .



Then the transfer function  $G_c(s)$  of the controller is

$$G_c(s) = \frac{K_i}{s}$$

The integrator will function as follows:

- If the error  $e(t)$  is positive, that is  $r(t) > y(t)$ , the controller output (and hence the torque produced by the motor) will *increase* at a rate proportional to the error.
- Similarly, if  $e(t) < 0$ , the controller output will *decrease* at a rate proportional to the magnitude of the error.
- If the error is zero, the integrator output will be *maintained at a constant value*.

The result is that the integrator will *continually adjust the motor torque so as to drive the error to zero*, at which point the supplied torque remains constant.

Assume that our *plant* (comprising the Power Amp, Rotational Plant and Tachometer) has a transfer function

$$G_p(s) = \frac{V_t(s)}{V_c(s)} = \frac{K_a K_m K_t / N}{Js + B}$$

Then the forward loop transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{K_i}{s} \cdot \frac{K_a K_m K_t / N}{Js + B}$$

The closed-loop transfer function is

$$G_{cl}(s) = \frac{V_t(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{K_i K_a K_m K_t / N}{Js^2 + Bs + K_i K_a K_m K_t / N}$$

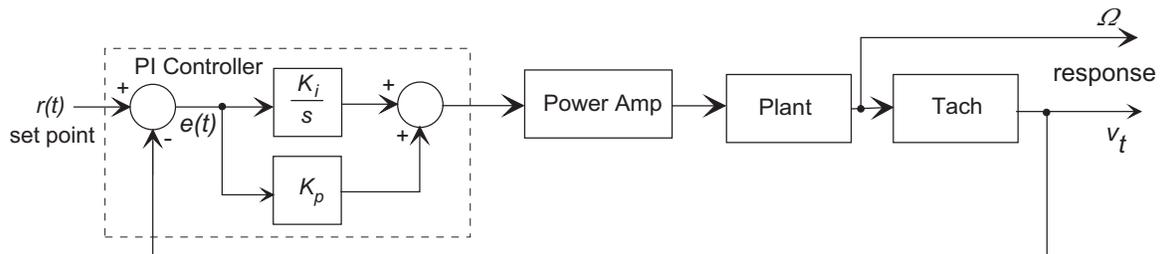
For a step input  $r(t) = A$ , the final-value theorem states

$$\begin{aligned} \lim_{t \rightarrow \infty} v_t(t) &= \lim_{s \rightarrow 0} (sV_t(s)) = \lim_{s \rightarrow 0} \left( \frac{A}{s} \frac{K_i K_a K_m K_t / N}{Js^2 + Bs + K_i K_a K_m K_t / N} \right) \\ &= A \end{aligned}$$

Note that the system has now become second order, and that the steady-state error will be zero.

**Proportional plus Integral (PI) Control:** Pure integral control is rarely used in practice, and you will see why in the course of this lab. PI control, on the other hand, is used very often. In PI control, the controller uses a linear combination of proportional and integral control actions:

$$\begin{aligned} G_c(s) &= K_p + K_i \frac{1}{s} \\ &= \frac{K_p s + K_i}{s} \\ &= K_p \left( \frac{s + K_i / K_p}{s} \right) \end{aligned}$$



The *plant* transfer function (comprising the Power Amp, Rotational Plant and Tachometer) is

$$G_p(s) = \frac{V_t(s)}{V_c(s)} = \frac{K_a K_m K_t / N}{Js + B}$$

The forward loop transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{K_p s + K_i}{s} \cdot \frac{K_a K_m K_t / N}{Js + B}$$

The closed-loop transfer function is

$$\begin{aligned} G_{cl}(s) &= \frac{V_t(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \\ &= \frac{(K_p K_a K_m K_t / N) s + K_i K_a K_m K_t / N}{J s^2 + (B + K_p K_a K_m K_t / N) s + K_i K_a K_m K_t / N} \end{aligned}$$

For a step input  $r(t) = A$ , the final-value theorem states

$$\begin{aligned} \lim_{t \rightarrow \infty} v_t(t) &= \lim_{s \rightarrow 0} (s V_t(s)) = \lim_{s \rightarrow 0} \left( \frac{A}{s} \frac{(K_p K_a K_m K_t / N) s + K_i K_a K_m K_t / N}{J s^2 + (B + K_p K_a K_m K_t / N) s + K_i K_a K_m K_t / N} \right) \\ &= A \end{aligned}$$

so that again, the steady-state error is zero. PI control eliminates steady-state error, just as does pure I control, but the additional constant  $K_p$  enables the system's damping to be specified.

We note in passing that I control has introduced an open-loop pole at the origin ( $s = 0$ ), and that PI control has introduced a pole at the origin, and an open-loop zero at  $s = -K_i/K_p$ .

## Appendix B: The Plant Transfer Function

In previous labs we found the plant transfer function to be

$$G_p(s) = \frac{V_t(s)}{V_c(s)} = \frac{K_a K_m K_t / N}{J_{eq} s + B_{eq}}$$

where  $V_t(s)$  is the tachometer output voltage, and  $V_c(s)$  is the controller output (input to the power amplifier), and we have measured or calculated the following numbers:

- $J_{eq} = 0.03 N.m^2$
- $B_{eq} = 0.014 N.m.s/rad$  (lab average)
- $K_a = 2.0 A/v$
- $K_m = 0.0292 N.m/A$  (lab average)
- $K_t = (0.016 \frac{v}{rev/min}) (60 \frac{s}{min}) (\frac{1 rev}{2\pi rad}) = 0.153 \frac{v}{rad/s}$
- $N = \frac{44}{180} = 0.244$