2.004 MODELING DYNAMICS AND CONTROL II Spring 2003

Solutions to Problem Set No. 3

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Problem 1 Impacts between two equal masses and one larger mass. In this problem there are three finite masses and three two-mass collisions. The generic results in Eq.(1) of Problem 1 can be used to determine the post-impact velocities for each colllision. The space-time diagram for the sequence of impacts is shown in Fig. 3.

Note that in the first and third impacts the collisions involve a moving mass striking a stationary mass of *equal* mass. In Problem 1 it was pointed out that in this special case there is a pure transfer of momentum from the impacting mass to the impacted mass. After the first collision the impacting mass which had velocity v_o to the right suddenly stands still while the stationary impacted mass suddenly begins to move with the same velocity v_o. A similar transfer of velocity occurs in the third impact where the impacting mass has the velocity v_1 to the left.

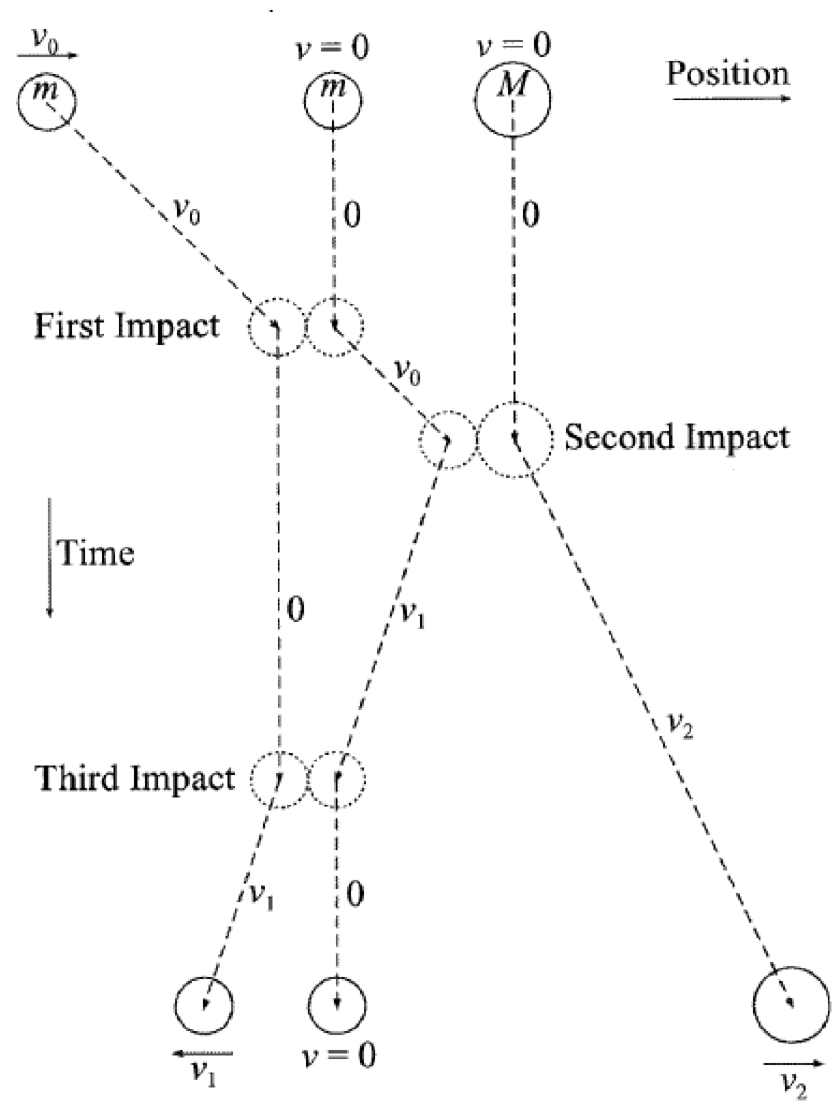


Figure 3: Space-time diagram of sequence of impacts.

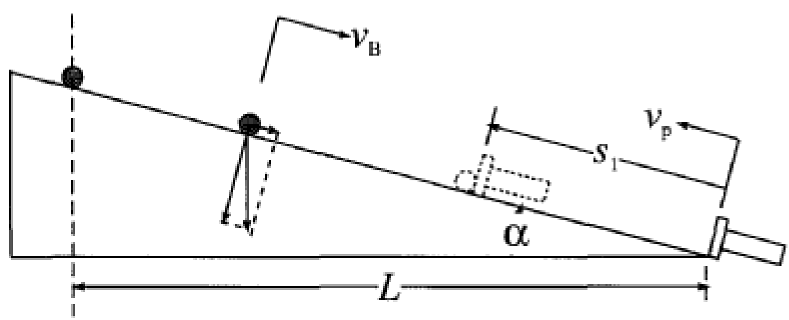
The second impact in Fig. 3 represents a collision of exactly the same form as the generic collision described in problem 1. A mass m with velocity v_o to the right strikes a larger stationary mass M. The velocities immediately after the impact are given by Eq.(1) of Problem 1. As indicated in Fig. 3, the smaller impacting mass rebounds to the left with a velocity of magnitude v_1, where by analogy with Eq.(1)

$$v_1 = \frac{M - m}{M + m} v_o$$

and the larger impacted mass suddenly acquires a velocity of magnitude v_2 to the right, where by analogy with Eq.(1),

$$v_2 = \frac{2m}{M + m} v_o$$

Problem 2. At $t = 0$ the piston starts uphill at the constant speed v_p, and the ball starts downhill with initial velocity zero. Because of gravity the ball accelerates down the slope with acceleration $g \sin \alpha$. At time t_1 there is an impact as the ball hits the piston. It is assumed that the piston velocity does not change during the impact, but shortly afterward the piston is stopped so that there is no second impact before the ball reaches its highest elevation h_{max}. To solve this problem it is necessary to determine the time (and place) of the collision, and then to solve the impact problem to find the new velocity of the ball, which fixes the kinetic energy of the ball immediately after the impact. Finally, it is necessary to determine the height h_{max} at which this kinetic energy is totally converted to potential energy.



Let distance up the slope be denoted by s. The piston starts at $s = 0$ with constant velocity v_p. The displacement of the piston at time t, before it hits the ball, is

$$s_p = v_p t$$

The ball starts at $s = L/\cos\alpha$ and accelerates down the slope with acceleration $-g\sin\alpha$. Integrating the acceleration yields

$$v_B = -gt\sin\alpha \qquad \text{and} \qquad s_B = \frac{L}{\cos\alpha} - \frac{1}{2}gt^2\sin\alpha$$

The collision occurs at time t_1, when $s_B = s_p$

$$\frac{L}{\cos\alpha} - \frac{1}{2}gt_1^2\sin\alpha = v_pt_1 \qquad \text{or} \qquad t_1^2 + 2\frac{v_p}{g\sin\alpha}t_1 - \frac{2L}{g\sin\alpha\cos\alpha}$$

This is a quadratic equation for the the time t_1 of collision. The positive root of this equation is

$$t_1 = \frac{v_p}{g\sin\alpha}\left(\sqrt{1 + \frac{2gL\sin\alpha}{v_P^2\cos\alpha}} - 1\right) \tag{1}$$

This is the answer to part (a) of Problem 2. The location where the collision occurs is at $s = s_1$, where

$$s_1 = v_pt_1 = \frac{v_p^2}{g\sin\alpha}\left(\sqrt{1 + \frac{2gL\sin\alpha}{v_P^2\cos\alpha}} - 1\right)$$

Next, it is necessary to solve the impact broblem. Just before the collision the piston velocity is v_p and the ball velocity is $-gt_1\sin\alpha$. The relative velocity of approach is $v_p + gt_1\sin\alpha$. Immediately after the collision the piston still has the velocity v_p, but the ball has a new velocity v_B up the slope, so the relative velocity of separation is $v_B - v_p$. Now the coefficient of restitution e, is the ratio of the relative velocity of separation to the relative velocity of approach, which implies

$$v_B - v_p = e(v_p + gt_1\sin\alpha) \qquad \text{or} \qquad v_B = (1+e)v_p + egt_1\sin\alpha$$

(b) Immediately after the impact the ball is at the elevation $h_1 = s_1\sin\alpha$ and has the velocity v_B just obtained. After the impact, the ball slides up the slope, without friction, exchanging kinetic energy for additional potential energy.

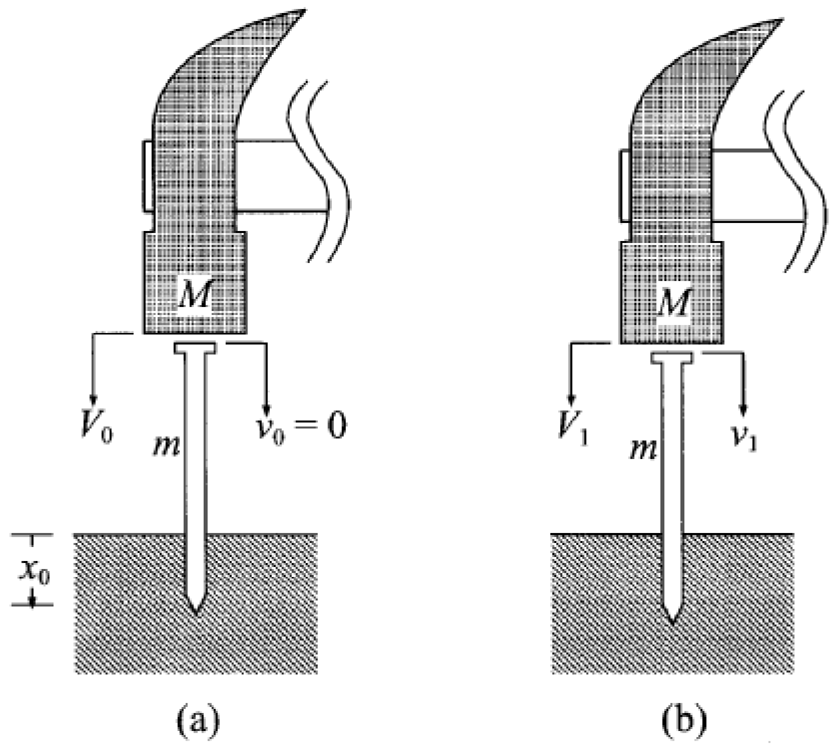
$$\frac{1}{2}mv_B^2 = mg(h_{max} - h_1)$$

The maximum elevation reached is

$$h_{max} = v_pt_1\sin\alpha + \frac{1}{2g}[(1+e)v_p + egt_1\sin\alpha]^2$$

where t_1 is given by Eq.(1).

Problem 3. The hammerhead strikes the nail with velocity V_0 as shown in (a) below. After the impact the nail has velocity v_1 and the hammerhead has the velocity V_1 as shown in (b) below.



(a) The initial momentum of the hammerhead is MV_0 in the downward direction and the combined downward momentum of the hammer and the nail, after the impact, is

$MV_1 + mv_1$. Conservation of momentum requires

$$MV_1 + mv_1 = MV_0$$

(b) In the experiment, the nail falls through a distance H and acquires a downward velocity $v_o = \sqrt{2gH}$. It then impacts the motionless hammerhead and rebounds with an upward velocity $v_1 = \sqrt{2gh}$. The downward velocity V_1 of the hammerhead, immediadely after the impact was not measured. It can, however, be recovered from the measurements of v_o and v_1 by applying the conservation of downward momentum

$$mv_o = MV_1 - mv_1 \qquad \text{or} \qquad V_1 = \frac{m}{M}(v_o + v_1)$$

The coefficient of restitution e is the ratio of the relative velocity of separation after the impact, $v_1 + V_1$, to the relative velocity of approach v_o

$$e = \frac{v_1 + V_1}{v_o} = \frac{v_1}{v_o}\left[1 + \frac{m}{M}(1 + \frac{v_o}{v_1})\right] = \sqrt{\frac{h}{H}}\left[1 + \frac{m}{M}\left(1 + \sqrt{\frac{H}{h}}\right)\right]$$

When the given data is inserted in this equation the result is

$$e = \sqrt{0.1}[1 + 0.01(1 + \sqrt{10})] = 0.329$$

When the hammer hits the nail, the relative velocity of approach is V_0, and the relative velocity of separation is $v_1 - V_1$, so the second equation is

$$v_1 - V_1 = eV_0$$

with $e = 0.329$.

(c) The equations in (a) and (b) are two simultaneous equations for the velocities v_1 and V_1 after the impact.

$$\begin{bmatrix} m/M & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} v_1 \\ V_1 \end{Bmatrix} = \begin{Bmatrix} V_0 \\ eV_0 \end{Bmatrix}$$

The solution for v_1 is

$$v_1 = \frac{1+e}{1+m/M} V_0 = \frac{1+0.329}{1+0.01} 10 = 13.16 \text{ m/s}$$

(d) The equation of motion for the nail under linear drag is

$$m\frac{dv}{dt} + bv = 0$$

where $v = dx/dt$ and x is measured downward into the block of wood. The solution for v, starting from $v = v_1$ at $t = 0$ is $v = v_1 \exp\{-bt/m\}$. To get the additional penetration δ, the equation $v = dx/dt$ is integrated over infinite time from x_o to $x_o + \delta$

$$\delta = \int_{x_o}^{x_o+\delta} dx = \int_0^{\infty} v_1 \exp\{-bt/m\}\, dt = v_1 \frac{m}{b} \qquad \text{or} \qquad \delta = \frac{m}{b} \frac{1+e}{1+m/M} V_0$$

(e) The viscous drag coefficient b is obtained by inserting the given data into the previous formula.

$$b = \frac{v_1 m}{\delta} = \frac{13.16 \times 0.005}{0.002} = 32.9 \text{ N/m/s}$$

(f) When the penetration resistance is $R = \mu_k n x$, the equation of motion for the nail is

$$m\frac{dv}{dt} + \mu_k n x = 0$$

Since the resistance is a function of x, it is convenient to consider that the velocity v is also a function of x and to use the chain rule to write

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx} v = \frac{d}{dx}(\frac{v^2}{2})$$

Then the equation of motion becomes

$$m\frac{d}{dx}(\frac{v^2}{2}) + \mu_k n x = 0$$

which can be integrated from $x = x_o$, where $v = v_1$, to $x = x_o + \delta$, where $v = 0$, to get

$$m\int_{v_1}^{0} \frac{v_1^2}{2} + \mu n \int_{x_o}^{x_o+\delta} x\,dx = -m\frac{v_1^2}{2} + \mu n \left[\frac{(x_o+\delta)^2}{2} - \frac{x_o^2}{2}\right] = 0$$

This result can be written as an equation for finding δ if the other parameters are known

$$\delta^2 + 2\delta x_o - \frac{m}{\mu_k n} v_1^2 = 0 \qquad (1)$$

or, as an equation for finding $\mu_k n$, if, as is the case here, the other parameters are known

$$\mu_k n = \frac{m v_1^2}{\delta(2x_o + \delta)}$$

Inserting the given data, we find

$$\mu_k n = \frac{0.005(13.16)^2}{0.002(2(0.01) + 0.002)} = 19,690 \text{ N/m}$$

(g) With all other factors the same, the hammer is applied with half the original velocity V_0. From (c) above it is seen that the initial velocity v_1 of the nail is also cut in half. In the linear resistance model, the additional penetration δ is simply proportional to the v_1, so cutting V_0 in half, means cutting δ in half, so the new $\delta = 2$ mm$/2 = 1$ mm.

(h) For the Coulomb friction model, the new δ is found by solving Eq.(1). Here it is seen that δ depends non-linearly on v_1. The solution of the quadratic equation (1) is

$$\delta = -x_o \pm \sqrt{x_o^2 + \frac{m v_1^2}{\mu_k n}}$$

Since δ must be positive, the positive square root must be used. Inserting the data with $v_1 = 13.16/2 = 6.58$ m/s, yields

$$\delta = -0.01 + \sqrt{0.0001 + \frac{(0.005)(6.58)^2}{19,690}} = 0.000536 \text{ m}$$

This is 0.536 mm, roughly half of the prediction made by the linear model.