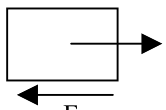
Problem Set 1 Solutions

1.



$$m\frac{dv_m}{dt} = f_{friction} = -Bv_m$$

$$v_m = v_0 e^{-\frac{B}{m}t}$$

$$\text{Time constant} = \frac{m}{B}$$

$$m = 10kg,\ v_m = \frac{1}{2}v_0\ at\ t = 5\sec$$

$$\frac{1}{2}v_0 = v_0 e^{-\frac{B}{10}5}$$

$$B = 1.386$$

2.

$$T_{friction} = \frac{d}{2} f_{friction} = \frac{d}{2} B \frac{d}{2} \Omega = B \frac{d^2}{4} \Omega = K_B \Omega, \text{ where } K_B = B \frac{d^2}{4}$$

$$T_{input} = K_i I_s(t), \text{ where } K_i \text{ is constant}$$

$$T_{input} - T_{friction} = J \frac{d\Omega}{dt}$$

$$J \frac{d\Omega}{dt} + K_B \Omega = K_i I_s(t)$$

The input I_s can be represented as

$$I_s(t) = \frac{I_m}{T} t - \frac{I_m}{T} (t - T) u(t - T)$$

Consider the Laplace transform, $L[f(t-T)1(t-T)] = e^{-Ts} F(s)$.

Using Laplace Transform,

$$I_s(s) = K_i \frac{I_m}{T} \frac{1}{s^2} [1 - e^{-Ts}]$$

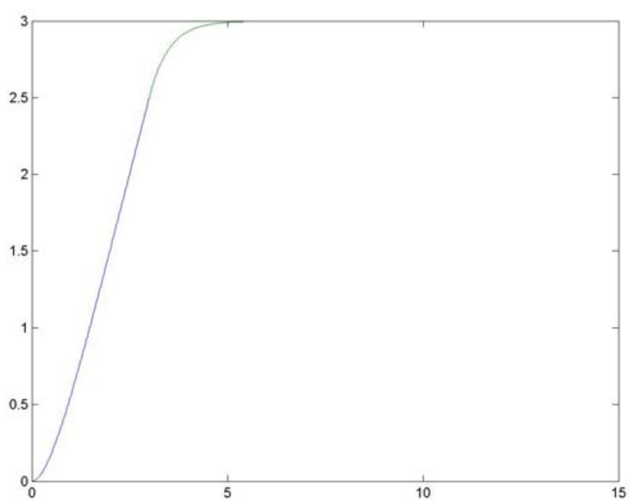
$$\Omega(s) = K_i \frac{I_m}{T} \frac{1}{s^2 (Js + K_B)} [1 - e^{-Ts}]$$

Using partial fraction and the inverse Laplace Transform, we can get

$$\Omega(t) = \frac{K_i I_m}{T K_B} [(-\frac{J}{K_B} + t + \frac{J}{K_B} e^{-\frac{K_B}{J} t}) - (-\frac{J}{K_B} + t - T + \frac{J}{K_B} e^{-\frac{K_B}{J}(t-T)}) u(t - T)]$$

Time constant is $\frac{J}{K_B}$.

Based on the above equation, we can sketch a graph with time constant =0.5.



3.

$$\tau_s \frac{d^2\varphi}{dt^2} + \frac{d\varphi}{dt} = K_v v_1$$

$$v_\varphi = k_\varphi \varphi$$

$$v_1 = K(v - v_\varphi)$$

a) Once substitute all the equations,

$$\tau_s \frac{d^2\varphi}{dt^2} + \frac{d\varphi}{dt} + K_v K k_\varphi \varphi = K_v K v$$

b) At steady state,

$$\frac{d^2\varphi}{dt^2} = 0 \ and \ \frac{d\varphi}{dt} = 0$$

$$\frac{\varphi}{v} = \frac{1}{k_\varphi}$$

c)

$$\frac{\varphi(s)}{v(s)} = \frac{K_v K}{\tau_s s^2 + s + K_v K k_\varphi} = \left(\frac{1}{k_\varphi}\right)\frac{K_v K k_\varphi/\tau_s}{s^2 + (1/\tau_s)s + (K_v K k_\varphi/\tau_s)}$$

$$w_n{}^2 = K_v K k_\varphi/\tau_s,\ w_n = \sqrt{K_v K k_\varphi/\tau_s}\ :\ undamped\ natural\ frequency$$

$$2\xi w_n = 1/\tau_s,\ \xi = \frac{1}{2 w_n \tau_s} = \frac{1}{2\tau_s\sqrt{K_v K k_\varphi/\tau_s}}\ :\ damping\ ratio$$

d)

At critical damping, there is no overshoot and fastest. So, ξ *should be* 1.

$$\xi = \frac{1}{2 w_n \tau_s} = \frac{1}{2\tau_s\sqrt{K_v K k_\varphi/\tau_s}} = 1$$

$$K = \frac{1}{4K_v k_\varphi \tau_s}$$

$$\varphi(t) = \frac{1}{k_\varphi}[1 - e^{-w_n t}(1 + w_n t)]$$

$$w_n = \sqrt{K_v K k_\varphi/\tau_s} = 1/\tau_s,\ when\ K = \frac{1}{4K_v k_\varphi \tau_s}$$

$$(\varphi)_{t\to\infty} = \frac{1}{k_\varphi}$$

$$0.9 = 1 - e^{-w_n T}(1 + w_n T) = 1 - e^{-\frac{T}{\tau_s}}\left(1 + \frac{T}{\tau_s}\right)$$

From the above equation, we can calculate T. $T/\tau_s \approx 3.9$