2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

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Spring Term 2008

Solution of Problem Set 6

Assigned: March 28, 2008 Due: April 4, 2008

Problem 1:

Note that for part a and c we have a zero-pole cancellation in LHP or origin. However, the MATLAB does not recognize this cancellation and we have to manually deal with it. The poles/zeros are extracted by *zpkdata* command and s-plane plot is obtained by *pzplot* command.

(a)

$$\frac{dy}{dt} + 3y = \frac{du}{dt} + 3u \Rightarrow G = \frac{s+3}{s+3} = 1$$

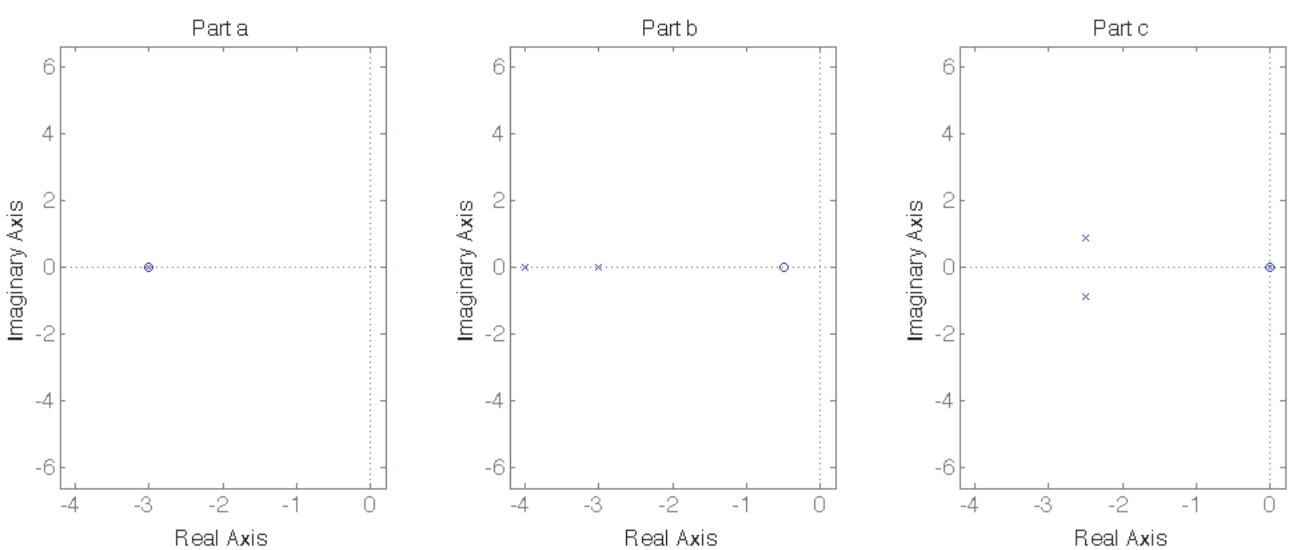
(b)

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = 2\frac{du}{dt} + u \Rightarrow G = \frac{2s+1}{s^2+7s+12} = 2\frac{(s+0.5)}{(s+3)(s+4)}$$

(c)

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} = \frac{du}{dt} \Rightarrow$$

$$G = \frac{s+0}{s^3+5s^2+7s+0} = \frac{1}{s^2+5s+7} = \frac{1}{(s+2.5)^2+0.75} = \frac{1}{(s+2.5)^2+(\frac{\sqrt{3}}{2})^2}$$



MATLAB Command – line :

```
>> [Z,P,K]=zpkdata(tf([1 3],[1 3]))   % Part a

Z =

    [-3]


P =

    [-3]


K =

     1

>> [Z,P,K]=zpkdata(tf([2 1],[1 7 12])), P=P{:}   % Part b

Z =

    [-0.5000]


P =

    [2x1 double]


K =

     2


P =

    -4
    -3

>> [Z,P,K]=zpkdata(tf([1 0],[1 5 7 0])), P=P{:}   % Part c

Z =
```

```
    [0]


P =

    [3x1 double]


K =

     1


P =

         0
   -2.5000 + 0.8660i
   -2.5000 - 0.8660i
```

Problem 2:

(a) Having a steady-state value means that the system is stable. Consequently, no poles can be in RHP (right half plane). This condition is satisfied by all options. Furthermore, the steady-state unit step response is equal to $G(0)$. If there is a zero at the origin then $G(0) = 0 \Rightarrow y_{ss} = 0$, which eliminates option c. Besides, if there is a pole at the origin then $G(0) = \frac{1}{0} = \infty \Rightarrow y_{ss} = \infty$ (y_{ss} is unbounded), which eliminates option b and d. Hence, the only feasible option is option a.

(b)

$$G(s) = K\frac{(s-1)}{(s-(-2))(s-(-5))} = K\frac{(s-1)}{(s+2)(s+5)}$$

$$y_{ss} = G(0) \Rightarrow 2 = K\frac{(0-1)}{(0+2)(0+5)} \Rightarrow K = -20 \Rightarrow G(s) = 20\frac{1-s}{(s+2)(s+5)}$$

(c) Modal functions correspond to e^{-2t} and e^{-5t}:

$$G(s) = 20(\frac{1}{s+2} - \frac{2}{s+5}) \Rightarrow g(t) = 20(e^{-2t} - 2e^{-5t})$$

Problem 3:

(a) Since the system, $G(s)$, has a zero at $s = z$, we can say that $G(s) = G_1(s)(s-z)$ such that $G_1(s)$ has no pole at s=z . The forced response can be computed from $Y(s) = G(s)U(s) = G(s)\frac{A}{s-z} = G_1(s)(s-z)\frac{A}{s-z} = AG_1(s)$. This means that $Y(s)$ has no pole at $s = z$ as well. Consequently particular solution (corresponding to a pole at $s = z$) is zero.

(b)

$$G(s) = G_1(s)(s-z_1)(s-z_2) = G_1(s)(s-(-\sigma+j\omega))(s-(-\sigma+j\omega)) = G_1(s)((s+\sigma)^2+\omega^2))$$

Again $G_1(s)$ has no pole corresponding to complex conjugate pair of $-\sigma \pm j\omega$.

$$u(t) = Ae^{-\sigma t}\sin(\omega t + \phi) = Ae^{-\sigma t}(\sin(\omega t)\cos(\phi) + cos(\omega t)\sin(\phi))$$

$$U(s) = A\frac{\omega\cos(\phi) + (s+\sigma)\sin(\phi)}{(s+\sigma)^2 + \omega^2}$$

$$Y(s) = G(s)U(s) = AG_1(s)(\omega\cos(\phi) + (s+\sigma)\sin(\phi))$$

Consequently, $Y(s)$ has no pole at complex conjugate pair of $-\sigma \pm j\omega$ as well. Hence the forced response is zero due to zero-pole cancellation.

Problem 4:

(a)

$$u(t) = e^{-bt} \Rightarrow U(s) = \tfrac{1}{s+b}$$
$$Y(s) = H(s)U(s) = \tfrac{a}{(s+a)(s+b)} = \tfrac{a}{a-b}(\tfrac{1}{s+b} - \tfrac{1}{s+a})$$
$$y(t) = \tfrac{a}{a-b}(e^{-bt} - e^{-at})$$

(b) $H(s)|_{s=-a} = \frac{a}{0} = \infty$ and $Y(s)|_{s=-a} = \frac{a}{0\times(b-a)} = \infty$. The particular solution is equal to $y_p(t) = \frac{a}{a-b}e^{-bt}$ and $\lim_{b\to a} y_p(t) = \infty$.

(c) The overall system response is yet bounded when $b = a$. This can be computed by below limit from L'Hospital's rule for $\frac{0}{0}$ limit:

$$\lim_{b\to a} y(t) = \lim_{b\to a}\frac{a}{a-b}(e^{-bt} - e^{-at}) = a\lim_{b\to a}\frac{\frac{\partial(e^{-bt}-e^{-at})}{\partial b}}{\frac{\partial(a-b)}{\partial b}} = a\lim_{b\to a}\frac{-te^{-bt}}{-1} = ate^{-bt}$$

Problem 5:.

(a)

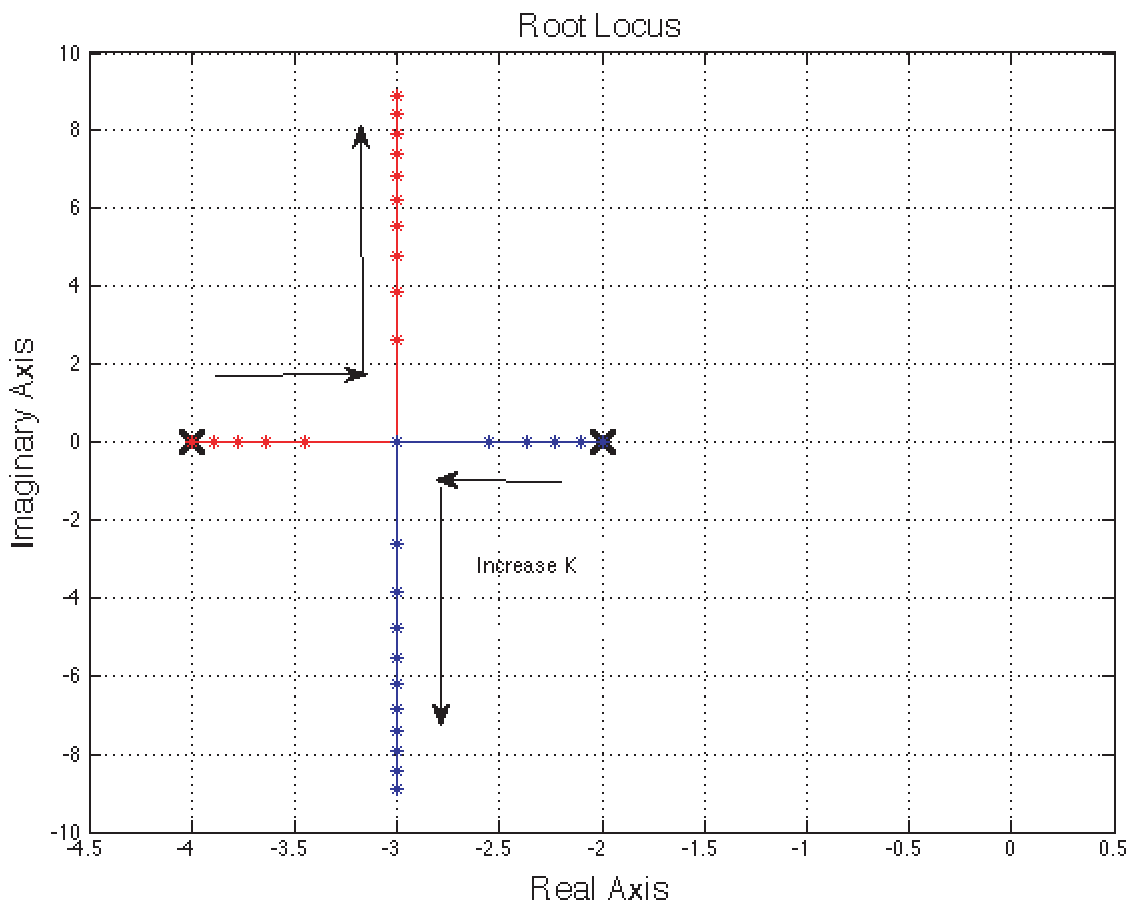
$$G_p(s) = K\frac{1}{(s+2)(s+4)}$$

$$\text{Unity Gain} \Rightarrow G(0) = 1 \Rightarrow K = 8 \Rightarrow G_p(s) = \frac{8}{(s+2)(s+4)} = \frac{8}{s^2+6s+8}$$

(b)

$$G_{cl}(s) = \frac{K_pG_p(s)}{1+K_pG_p(s)} = \frac{8K_p}{s^2+6s+8+8K_p} = \frac{8K_p}{s^2+6s+8(1+K_p)}$$

(c-d) The closed-loop poles for the values of $K_p = 0, 0.025, 0.05, 0.075, 1, 1, 25, 1, 2, 3, \ldots, 10$ are plotted via attached code. Alternatively we could use *rlocus* command or *rltool* toolbox.



(e)

$$\text{Critically Damped} \Rightarrow \varsigma = 1 \Rightarrow\ 6 = 2\omega_n \Rightarrow \omega_n = 3$$

$$\omega_n^2 = 8(1 + K_p) = 9 \Rightarrow K_p = \frac{1}{8} = 0.125$$

MATLAB m − file :

```
clc, close all, clear
P_cl=[];
plot([-2 -4],[0 0],'k+'),hold on
for Kp=[0:.025:.125 1:10]

    [Z,P,K]=zpkdata(tf(8*Kp,[1 6 (1+Kp)*8]));
    P_cl(end+1,:)=P{:};

end

plot(real(P_cl(:,1)),imag(P_cl(:,1)),'r-*')
```

```
plot(real(P_cl(:,2)),imag(P_cl(:,2)),'b-*')
grid on, box on
title ('Root Locus','fontsize',16)
xlabel ('Real Axis','fontsize',16)
ylabel ('Imaginary Axis','fontsize',16)
axis([-4.5 0.5 -10 10])
------------------ Computed Poles by Varying Kp -------------------
  >> P_cl

P_cl =

  -4.0000            -2.0000
  -3.8944            -2.1056
  -3.7746            -2.2254
  -3.6325            -2.3675
  -3.4472            -2.5528
  -3.0000 + 0.0000i  -3.0000 - 0.0000i
  -3.0000 + 2.6458i  -3.0000 - 2.6458i
  -3.0000 + 3.8730i  -3.0000 - 3.8730i
  -3.0000 + 4.7958i  -3.0000 - 4.7958i
  -3.0000 + 5.5678i  -3.0000 - 5.5678i
  -3.0000 + 6.2450i  -3.0000 - 6.2450i
  -3.0000 + 6.8557i  -3.0000 - 6.8557i
  -3.0000 + 7.4162i  -3.0000 - 7.4162i
  -3.0000 + 7.9373i  -3.0000 - 7.9373i
  -3.0000 + 8.4261i  -3.0000 - 8.4261i
  -3.0000 + 8.8882i  -3.0000 - 8.8882i
```