2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

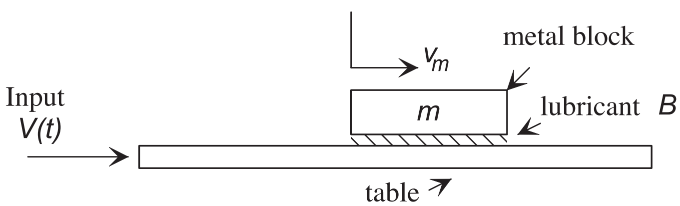
2.004 *Dynamics and Control II*

Spring Term 2008

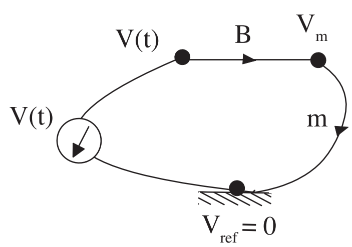
Solution of Problem Set 5

Assigned: March 7, 2008 Due: March 14, 2008

Problem 1:



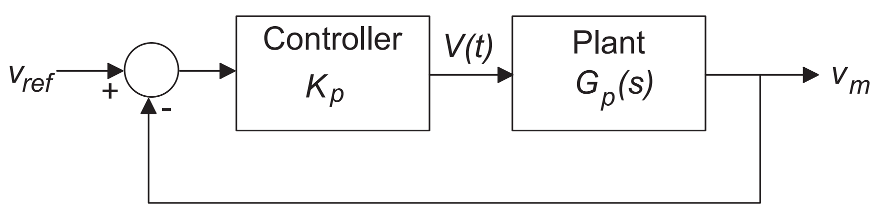
(a)



Use the voltage division rule:

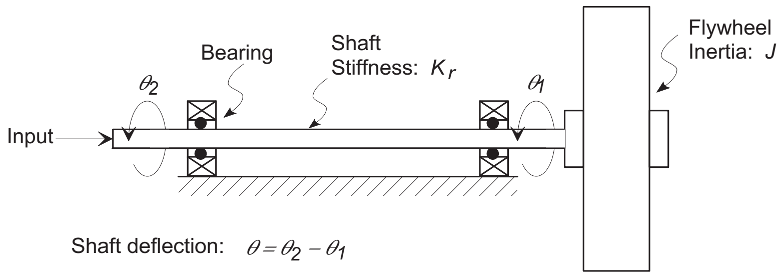
$$G_p(s) = \frac{V_m(s)}{V_(s)} = \frac{Z_m}{Z_m + Z_B} = \frac{\frac{1}{ms}}{\frac{1}{ms} + \frac{1}{B}} = \frac{B}{ms + B}$$

(b)



$$\frac{V_m(s)}{V_{ref}(s)} = \frac{K_p G_p(s)}{1 + K_p G_p(s)} = \frac{K_p B}{ms + (1 + K_p)B}$$

Problem 2: (Modified from Problem Set 4.)



All the calculations of this problem are carried out by attached MATLAB code.

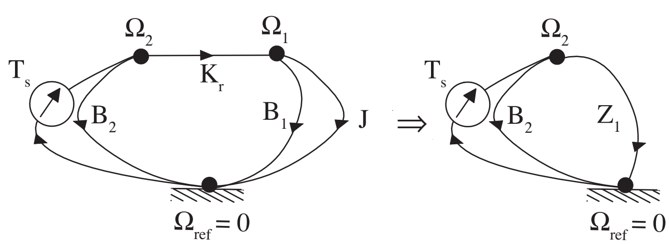
(a)

$$K_r = G\frac{\pi}{32}\frac{D_r^4}{l} = 1.02 \times 10^4 \ \frac{N.m}{rad}$$

$$J = \rho\frac{\pi}{32}\frac{D_f^4}{h} = 0.310 \ Kg.m^2$$

In the above relations h is the flywheel thickness and $D_f = .3 \ m$ and $D_r = .05 \ m$ are the flywheel/shaft diameters. The J of shaft can be similarly computed and found to be equal to $0.024 \ Kg.m^2$. The shaft inertia is an order of magnitude smaller than flywheel inertia and for our simple model we ignore it.

(b) Assume that $B_1 = B_2 = B$ and define $\dot{\theta}_1 = \Omega_1$ and $\dot{\theta}_2 = \Omega_2$:



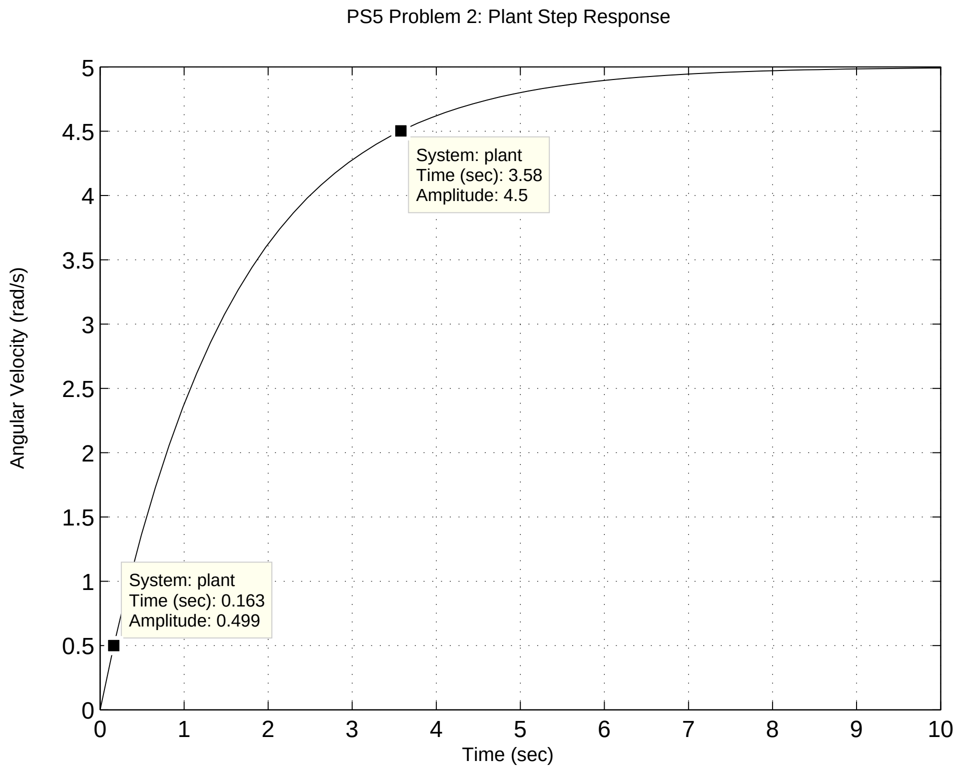
$$Z_1 = Z_{K_r} + (Z_{B_1}||Z_J) = \frac{s}{K_r} + \frac{1}{Js + B}$$

$$\text{Current Divider}: T_1 = T_s\frac{Y_1}{Y_1 + Y_{B_2}}$$

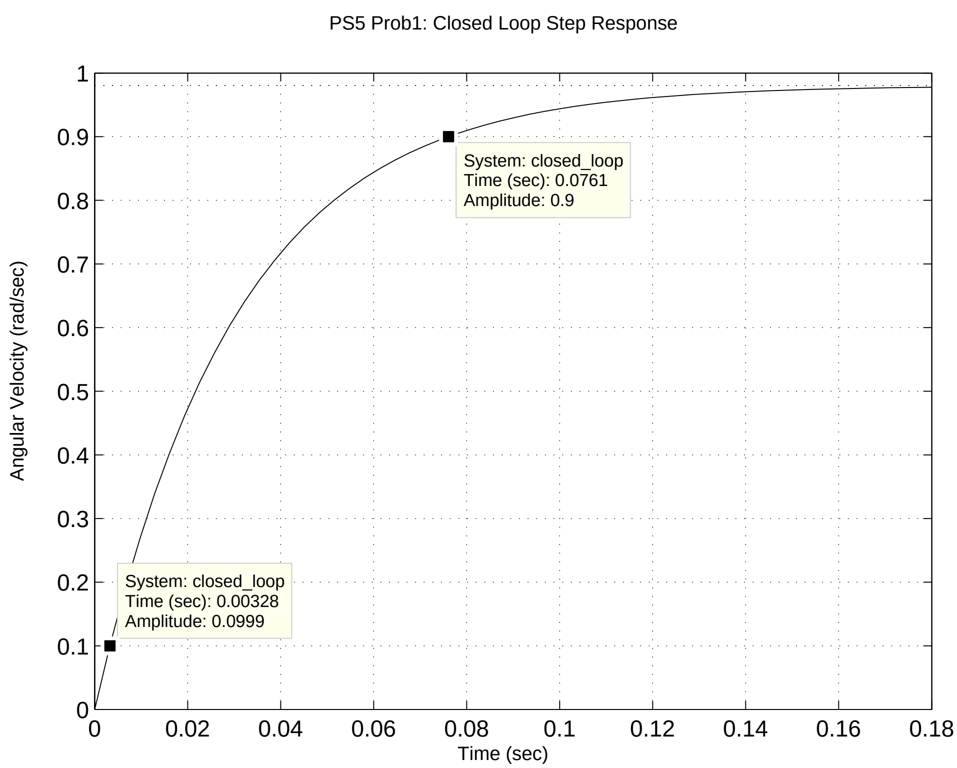
$$\Omega_1 = (Z_{B_1}||Z_J)T_1$$

$$G_p(s) = \frac{\Omega_1(s)}{T_s(s)} = \frac{K_r}{BJs^2 + (B^2 + K_rJ)s + 2K_rB} = \frac{1.019 \times 10^4}{0.0310s^2 + 3159s + 2037}$$

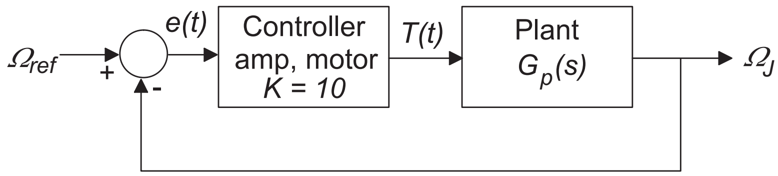
(c) For the open loop system the steady state output for a unit step is equal to $G(0) = \frac{1}{2B} = 5$. From this plot the "rise-time" is estimated to be equal to $3.58 - .163 = 3.427$ sec.



(d) From the plot the "rise-time" is estimated to be equal to $.0761 - .00328 = 0.0728$ sec.



(e) The proportional control has increased the rise time substantially. The rise time can be related to ω_n which can be computed from the transfer function:



$$\text{Closed Loop :} \quad \mathrm{G_p(s)} = \frac{\Omega_1(\mathrm{s})}{\mathrm{T_s(s)}} = \frac{\mathrm{K_r}}{\mathrm{BJs^2 + (B^2 + K_rJ)s + 2K_rB}} = \frac{1.019 \times 10^4}{0.0310\mathrm{s}^2 + 3159\mathrm{s} + 2037}$$

$$\text{Closed Loop :} \quad \frac{\Omega(\mathrm{s})}{\Omega_{\mathrm{ref}}(\mathrm{s})} = \frac{\mathrm{KG_p(s)}}{1 + \mathrm{KG_p(s)}} = \frac{\mathrm{KK_r}}{\mathrm{BJs^2 + (B^2 + K_rJ)s + 2K_rB + KK_r}}$$

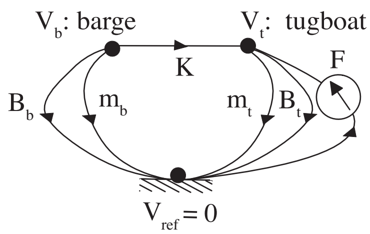
For the open loop system $w_n = \sqrt{\frac{2K_rB}{BJ}} = \sqrt{\frac{2K_r}{J}} = \sqrt{65685}\ \frac{rad}{s}$, while for the closed loop system $w_n = \sqrt{\frac{2K_rB + KK_r}{BJ}} = \sqrt{\frac{2K_r}{J} + \frac{KK_r}{BJ}} = \sqrt{65685 + 32843K}\ \frac{rad}{s}$ and hence K has a huge effect on the rise time.

MATLAB m – file :

```
clear all,clc,close all
%   Symbolic Transfer Function
syms B Kr J s Ts
Z1=s/Kr+(1/(J*s+B));Y1=1/Z1;YB=B;
T1=simple(Ts*Y1/(Y1+YB))
Omega_1=simple(T1*1/(J*s+B))
clear all
% Compute the value of J
d = 0.3; h = 0.05; rho = (7.8*1e-3)/1e-6; J = pi/32*rho*h*d^4
% Compute the value of K
G = 83e9; l = 5; D = 5e-2;  Kr = pi*G*D^4/32/l
% B=B1=B2
B=0.1;
% From our analysis the plant transfer function Gp(s)= WJ(s)/T(s) is:
%   Gp(s) = Kr/(B*J*s^2 + (Kr*J + B^2)s +2*Kr*B)
% Display the undamped natural freqency
wn = sqrt(Kr/J)
plant =tf([Kr],[B*J (Kr*J+B^2) 2*Kr*B])
figure(1)
step(plant);
ylabel('Angular Velocity (rad/s)')
title('PS5 Problem 2: Plant Step Response')
```

```
set(gcf,'color','w'),set(gca,'fontsize',12)
box on, grid on,set(gcf,'position',[100,100,600,400])
% Form closed loop system - controller, etc, gain is 10:
closed_loop = feedback(10*plant,1)
figure(2)
step(closed_loop)
ylabel('Angular Velocity (rad/sec)')
title('PS5 Prob1: Closed Loop Step Response')
set(gcf,'color','w'),set(gca,'fontsize',12)
box on, grid on,set(gcf,'position',[100,100,600,400])
```

Problem 3:



A linear graph model of the system is shown above and the node equations are derived below. The two masses represent inertial forces of the tugboat and the barge. Since usually m_t is very smaller than m_b, we can ignore the tugboat mass to further simplify the model. The K stiffness represents the cable elasticity and B_t and B_b represent water drags on tugboat and barge. This system has three independent energy storage elements and transfer functions like $\frac{V_t(s)}{F(s)}$ or $\frac{V_b(s)}{F(s)}$ are third orders. The attached MATLAB code computes the V_t and V_b and proves that our transfer functions would be third orders.

$$\begin{cases} \text{node (t)}: & F_{m_t} + F_{B_t} - F_K = F \\ \text{node (b)}: & F_{m_b} + F_{B_b} + F_K = 0 \end{cases}$$

$$\begin{cases} \text{node (t)}: & (Y_{m_t} + Y_{B_t})(V_t - 0) - Y_K(V_b - V_t) = F \\ \text{node (b)}: & (Y_{m_b} + Y_{B_b})(V_b - 0) + Y_K(V_b - V_t) = 0 \end{cases}$$

$$\begin{cases} \text{node (t)}: & (m_t s + B_t)V_t - \frac{K}{s}(V_b - V_t) = F \\ \text{node (b)}: & (m_b s + B_b)V_b + \frac{K}{s}(V_b - V_t) = 0 \end{cases}$$

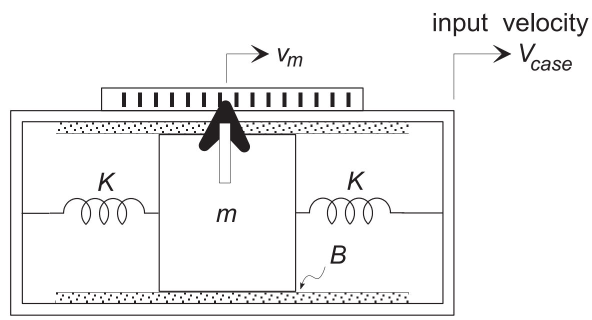
$$\begin{cases} \text{node (t)}: & (m_t s + B_t + \frac{K}{s})V_t - \frac{K}{s}V_b \quad = F \\ \text{node (b)}: & -\frac{K}{s}V_t + (m_b s + B_b + \frac{K}{s})V_b = 0 \end{cases}$$

MATLAB Command – line :

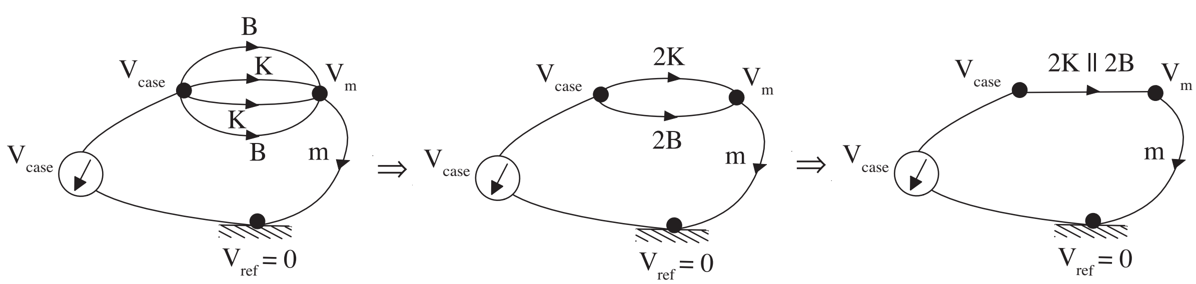
```
>> syms mt mb Bt Bb K F Vt Vb s real
>> eq1=(mt*s^2+Bt*s+K)*Vt-K*Vb-F*s;
```

```
>> eq2=(mb*s^2+Bb*s+K)*Vb-K*Vt;
>> [Vb,Vt]=solve(eq1,eq2,Vb,Vt)

Vb =

K*F/(mt*mb*s^3+s^2*mt*Bb+s*Bt*Bb+s*mt*K+s*K*mb+s^2*Bt*mb+Bt*K+K*Bb)



Vt =

(mb*s^2+Bb*s+K)*F/(mt*mb*s^3+s^2*mt*Bb+s*Bt*Bb+s*mt*K+s*K*mb+s^2*Bt*mb+Bt*K+K*Bb)
```

Problem 4:



(a) V_m is defined as the absolute velocity:



(b) There are two independent energy storage elements: one is the mass inertia (requiring to know V_m) and the other is the spring (requiring to know $x_m - x_{case}$). The two springs act in parallel and are dependent.

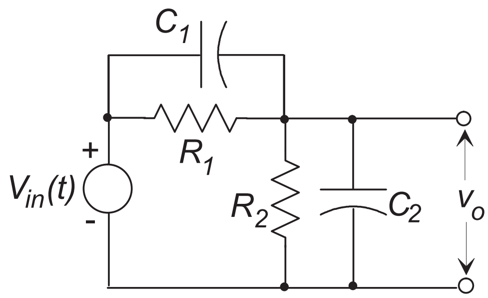
(c) The mesh equation for a single loop would be the same as the voltage division rule:

$$\text{Voltage Divider}: V_m = V_{case}\frac{Z_m}{Z_m + (Z_{2B}||Z_{2K})} = V_{case}\frac{\frac{1}{ms}}{\frac{1}{ms} + (\frac{\frac{s}{2K}\frac{1}{2B}}{\frac{s}{2K}+\frac{1}{2B}})}$$

$$V_m = V_{case}\frac{\frac{1}{ms}}{\frac{1}{ms}+\frac{s}{2Bs+2K}} = V_{case}\frac{2Bs+2K}{ms^2+2Bs+2K}$$

$$G(s) = \frac{x_m - x_{case}}{V_{case}} = \frac{V_m - V_{case}}{sV_{case}} = \frac{-ms}{ms^2+2Bs+2K}$$

Problem 5:



$$\text{Voltage Divder}: \frac{V_o(s)}{V_{in}(s)} = \frac{(Z_{R_2}||Z_{C_2})}{(Z_{R_1}||Z_{C_1})+(Z_{R_2}||Z_{C_2})}$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{\frac{R_2}{R_2C_2s+1}}{\frac{R_1}{R_1C_1s+1}+\frac{R_2}{R_2C_2s+1}} = \frac{R_2(R_1C_1s+1)}{R_1R_2(C_1+C_2)s+R_1+R_2}$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{s+20}{1.5s+30} = \frac{2}{3}$$

$$V_o(t) = \frac{2}{3}V_{in}(t)$$

We have two energy storage elements, but in general the order of system should be one. This means that our energy storage elements are dependent and that's true because if we know the voltage of one capacitor, the voltage of the other capacitor is know as well ($V_{in} = V_{C_1} + V_{C_2}$). Furthermore, for this particular given values of parameters the order of system is zero due to a zero-pole cancellation.