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2.004 Dynamics and Control II  
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## 2.004 Problem Set 1 Solutions

**Problem 1.** For the rotational system, we are given a flywheel spinning at rate  $\Omega(t)$ , driven by a torque  $T(t)$  that is opposed by a frictional torque  $T_f(t)$ . The governing differential comes from the torque balance,  $\sum M = J\dot{\Omega}$  – note that the sign of the frictional torque is opposite that of the driving torque:

$$T(t) - T_f(\Omega) = J\dot{\Omega}(t) \quad (1)$$

Collecting the  $\Omega$  terms:

$$J\dot{\Omega}(t) + T_f(\Omega) = T(t) \quad (2)$$

**Problem 2.** Given the above relation, we are asked to consider three separate cases for the frictional torque  $T_f(\Omega)$ :

- (a) Coulomb torque, *i.e.*,  $T_f(\Omega) = T_c \text{sgn}(\Omega)$ . If we assume unidirectional motion with  $\Omega > 0$  then  $T_f(\Omega) = T_c$ . Substituting into (2):

$$J\dot{\Omega}(t) = T(t) - T_c \quad (3)$$

- (b) Viscous torque, with  $T_f(\Omega) = B\Omega$ . Substituting into (2):

$$J\dot{\Omega}(t) + B\Omega(t) = T(t) \quad (4)$$

- (c) With both viscous and Coulomb torque, again substituting into (2):

$$J\dot{\Omega}(t) + B\Omega(t) = T(t) - T_c \quad (5)$$

**Problem 3.** We are asked to solve the above differential equations, given initial condition  $\Omega(0) = \Omega_0$  and no applied torque,  $T(t) = 0$ .

- (a) With  $T(t) = 0$ , substituting into (3):  $J\dot{\Omega}(t) = -T_c$ , or  $\dot{\Omega}(t) = -T_c/J$ . Integrating,

$$\Omega(t) = -\frac{T_c}{J}t + C$$

From the initial condition  $\Omega(0) = \Omega_0$ , we have  $\Omega_0 = C$ ; substituting into the above,

$$\Omega(t) = \Omega_0 - \frac{T_c}{J}t \quad (6)$$

- (b) From (4), with  $T(t) = 0$ ,

$$J\dot{\Omega}(t) + B\Omega(t) = 0. \quad (7)$$

Dividing through by  $J$ :

$$\dot{\Omega}(t) + \frac{B}{J}\Omega(t) = 0$$

Given the form of the above equation, we assume a solution of the exponential form,  $\Omega(t) = Ae^{-t/\tau}$ . Using the initial condition  $\Omega(0) = \Omega_0$ , we have  $\Omega_0 = A$ . Differentiating the assumed solution,  $\dot{\Omega}(t) = -(1/\tau)Ae^{-t/\tau}$ , so

$$-\frac{1}{\tau}Ae^{-t/\tau} + \frac{B}{J}Ae^{-t/\tau} = 0$$

Thus  $\tau = J/B$ , so we get:

$$\Omega(t) = \Omega_0 e^{-(\frac{B}{J})t} \quad (8)$$

(c) With both viscous and Coulomb torque, and  $T(t) = 0$ , from (5):

$$\begin{aligned} J\dot{\Omega}(t) + B\Omega(t) &= -T_c; & \Omega(0) &= \Omega_0 \\ \dot{\Omega}(t) + \frac{B}{J}\Omega(t) &= -\frac{T_c}{J}; & \Omega(0) &= \Omega_0 \\ \text{or } \frac{d\Omega(t)}{dt} + \frac{1}{\tau}\Omega(t) &= -\frac{T_c}{J}; & \text{where } \tau &\equiv \frac{J}{B} \end{aligned}$$

Separating variables:

$$\frac{d\Omega(t)}{\Omega(t) + T_c/B} = -\frac{1}{\tau}dt$$

Substituting in the above,  $\omega(t) = \Omega(t) + T_c/B$ ,  $d\omega(t) = d\Omega(t)$ :

$$\begin{aligned} \frac{d\omega}{\omega} &= -\frac{1}{\tau}dt \\ \ln \omega &= -\frac{t}{\tau} + C \\ \rightarrow \omega(t) &= e^{-t/\tau + C} \end{aligned}$$

Reversing the substitution of variables,

$$\begin{aligned} \Omega(t) &= \omega(t) - \frac{T_c}{B} \\ &= e^{-t/\tau + C} - \frac{T_c}{B} \\ &= Ae^{-t/\tau} - \frac{T_c}{B} \end{aligned}$$

To find the constant, we use the initial condition  $\Omega(0) = \Omega_0$ :

$$\begin{aligned} \Omega_0 &= A - \frac{T_c}{B} \\ A &= \Omega_0 + \frac{T_c}{B} \end{aligned}$$

So, finally:

$$\Omega(t) = \left( \Omega_0 + \frac{T_c}{B} \right) e^{-t/\tau} - \frac{T_c}{B} \quad (9)$$

$$\text{or, } \Omega(t) = \Omega_0 e^{-t/\tau} - \frac{T_c}{B} (1 - e^{-t/\tau}) \quad (10)$$

**Problem 4.** The moment of inertia of a solid cylinder of mass  $m$  and radius  $r$  about its axis is given by  $J = \frac{1}{2}mr^2$ , while the mass (assuming uniform density  $\rho$ ) is  $m = (\pi r^2 h)\rho$ , where  $h$  is the height of the cylinder. Thus the moment expands to

$$J = \frac{\pi}{2}(r^4 h)\rho \quad (11)$$

for a solid cylinder. For our flywheel, neglecting the spokes and hub we may consider just the outer ring (which given the  $r^4$  term above will dominate the inertia). Its moment of inertia, by superposition, is the difference between the moments of inertia of two cylinders of radius  $r_2$  and  $r_1$ , the outer and inner radii, respectively:

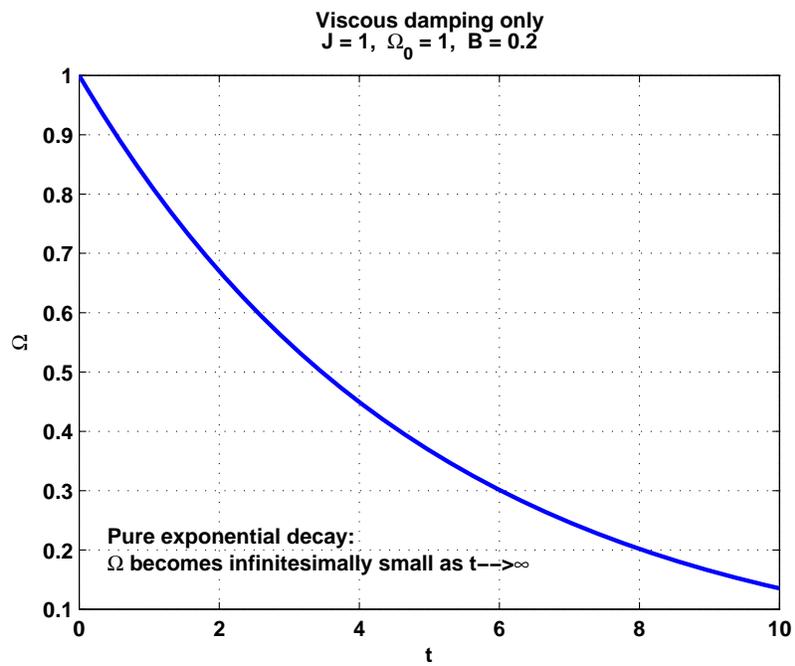
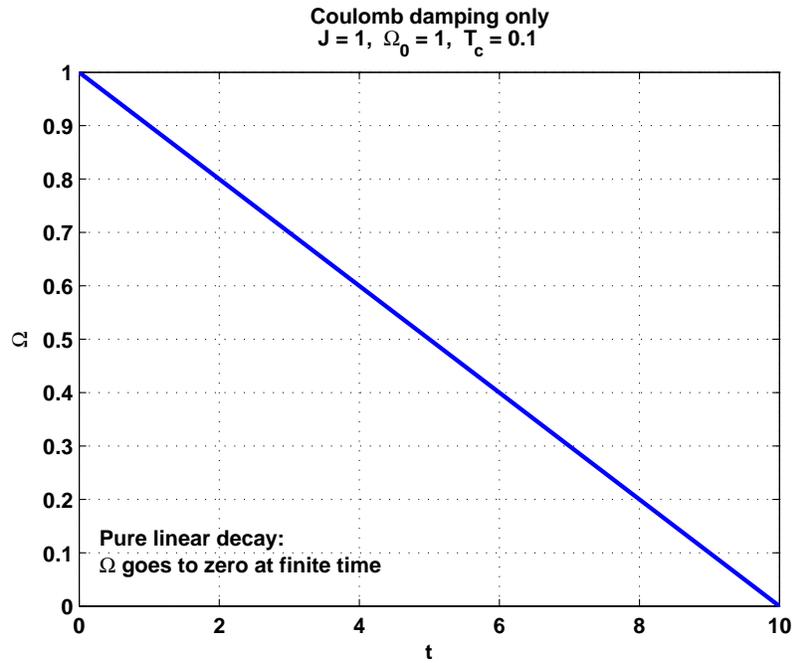
$$J = \frac{\pi}{2}(r_2^4 h)\rho - \frac{\pi}{2}(r_1^4 h)\rho = \frac{\pi h}{2}(r_2^4 - r_1^4)\rho \quad (12)$$

For the plant described in the lab handout,  $r_1 = 63.50 \times 10^{-3}$  m,  $r_2 = 104.8 \times 10^{-3}$  m,  $h = 22.23 \times 10^{-3}$  m, and  $\rho = 8230$  kg/m<sup>3</sup>. Thus,

$$J = \frac{\pi * 0.02223}{2}(0.1048^4 - 0.0635^4) * 8230 = 3.00 \times 10^{-2} \text{ kg m}^2 \quad (13)$$

**Problem 5.** From Problem 4, we know the form of the equation of motion for the flywheel with viscous damping, with coulomb damping, and with both forms of damping. Given motion data, we can take the general equation (from 4(c)) and attempt a least-squares fit to the experimental data, varying the coefficients  $T_c$  and  $B$  ( $\Omega_0$  is known; it is the measured velocity at  $t = 0$ , and  $J$  has been computed in Problem 4).

### Plots for Problem 3



Viscous + Coulomb damping  
 $J = 1, \Omega_0 = 1, B = 0.1, T_c = 0.1$

