2.003SC
Recitation 11 Notes: Double Pendulum System

EIGENVALUES and EIGENVECTORS

Consider the following matrix equation,

$$\underline{A}\underline{x} = \lambda \underline{x} \tag{1}$$

where $\underline{A}$ is a matrix of size nxn , $\underline{x}$ is a vector of length n, and λ is a scalar

For a given matrix, $\underline{A}$, the values of λ_i and $\underline{x}_i$, $i = 1, ..., n$ that satisfy the above equation are called (the matrix's) **eigenvalues** and **eigenvectors**, respectively.

Eigenvalues and eigenvectors are a very important and valuable concept that arises in many technical fields, especially vibrations.

Consequently, well-established, robust computational procedures exist for evaluating the eigenvalues and eigenvectors of a matrix.

Connection to Vibrations

Recall the matrix form of the equations of motion for an n-degree-of-freedom system,

$$\underline{M}\underline{\ddot{x}} + \underline{K}\underline{x} = 0$$

This can be re-written as

$$\underline{\ddot{x}} + \underline{M}^{-1}\underline{K}\underline{x} = 0$$

or

$$\underline{M}^{-1}\underline{K}\underline{x} = -\underline{\ddot{x}}$$

Recall that, for harmonic motion

$$\underline{\ddot{x}} = -\omega^2 \underline{x}$$

So, the matrix equation has the same form as (1) above, i.e. can be seen to be an eigenvalue problem.

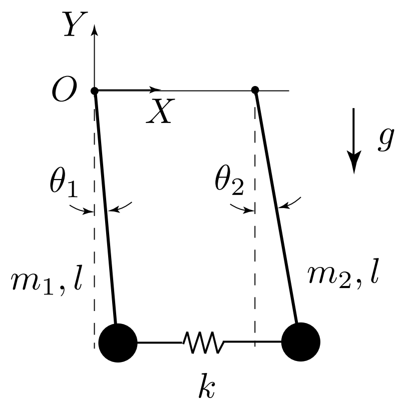
$$(\underline{M}^{-1}\underline{K})\underline{x} = (\omega^2)\underline{x}$$

where

- $\underline{A} = \underline{M}^{-1}\underline{K}$ is the system matrix
- the eigenvalues, λ_i , are the natural frequencies, ${\omega_i}^2$
- the eigenvectors, $\underline{x}_i$, are the natural modes

Double Pendulum System - Problem Statement

Consider a system of two masses and one spring as shown in the figure below. Note that θ_1 and θ_2 are small-angle displacements.



The system's equations of motion are

$$\ddot{\theta}_1 + \left(\frac{g}{l} + \frac{k}{m_1}\right)\theta_1 - \left(\frac{k}{m_1}\right)\theta_2 = 0$$

$$\ddot{\theta}_2 - \left(\frac{k}{m_2}\right)\theta_1 + \left(\frac{g}{l} + \frac{k}{m_2}\right)\theta_2 = 0$$

For the special case where $m_1 = m_2 = m$,

- Write the equations of motion in matrix notation.
- Find the characteristic equation
- Find the natural frequencies and natural modes

Double Pendulum System - Solution

EQUATIONS OF MOTION IN MATRIX NOTATION

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (\frac{g}{l} + \frac{k}{m_1}) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & (\frac{g}{l} + \frac{k}{m_2}) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

or

$$\underline{M\ddot{x}} + \underline{Kx} = 0$$

Setting $m_1 = m_2 = m$, the equations of motion are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (\frac{g}{l} + \frac{k}{m}) & -\frac{k}{m} \\ -\frac{k}{m} & (\frac{g}{l} + \frac{k}{m}) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

CHARACTERISTIC EQUATION

Assume the two masses undergo harmonic motion, i.e. they oscillate with the same frequency, ω, albeit different amplitudes, a_1, a_2.

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} cos(\omega t - \phi) \tag{3}$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -\omega^2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} cos(\omega t - \phi) \tag{4}$$

Substituting (3) and (4) into (2), we obtain,

$$\begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} cos(\omega t - \phi) + \begin{bmatrix} (\frac{g}{l} + \frac{k}{m}) & -\frac{k}{m} \\ -\frac{k}{m} & (\frac{g}{l} + \frac{k}{m}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} cos(\omega t - \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Dividing by $cos(\omega t - \phi)$, we obtain

$$\begin{bmatrix} -\omega^2 + (\frac{g}{l} + \frac{k}{m}) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + (\frac{g}{l} + \frac{k}{m}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

Setting the determinant equal to zero produces the CHARACTERISTIC EQUATION/POLYNOMIAL.

$$h^2 - 2h\omega^2 + \omega^4 - \left(\frac{k}{m}\right)^2 = 0 \qquad where \qquad h = \frac{g}{l} + \frac{k}{m}$$

NATURAL FREQUENCIES AND NATURAL MODES

Applying the quadratic formula to the characteristic equation,

$$\omega^2 = h \pm \frac{k}{m} = \left(\frac{g}{l} + \frac{k}{m}\right) \pm \frac{k}{m}$$

or

$$\omega_1 = \sqrt{\frac{g}{l}} \qquad ; \qquad \omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}}$$

From the first row of (5),

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) a_1 - \frac{k}{m} a_2 = 0$$

we can obtain the formula for the natural modes,

$$\frac{a_2}{a_1} = \frac{-m\omega^2 + 2k}{k}$$

which we evaluate at each of the natural frequencies,

$$\omega_1 = \sqrt{\frac{g}{l}} \qquad \rightarrow \qquad \frac{a_2}{a_1} = 1$$

$$\omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}} \qquad \rightarrow \qquad \frac{a_2}{a_1} = -1$$

General Solution

In general, however, (i.e. for arbitrary initial conditions), the system's free response will contain BOTH natural frequencies,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} cos(\omega_1 t - \phi_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} cos(\omega_2 t - \phi_2)$$

where A_1, A_2, ϕ_1, ϕ_2 are determined by initial conditions.

This general response can appear to be very irregular, with little discernible pattern. When the natural frequencies are close together, "beating" behavior can be observed.