

2.003 Engineering Dynamics

Problem Set 10 with solution

Problem 1

Figure 1. Cart with a slender rod

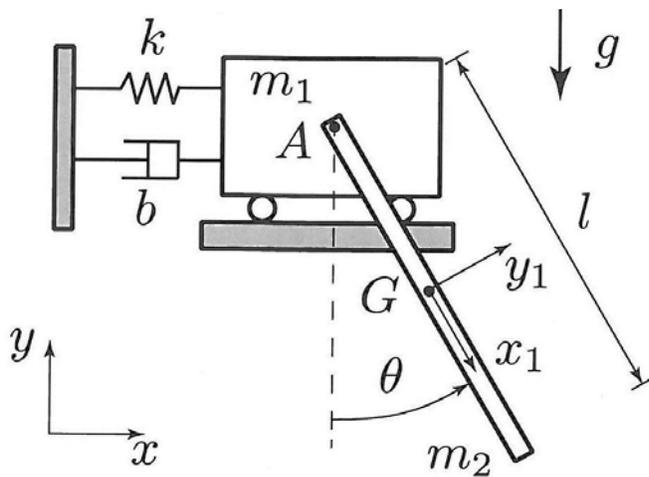


Figure 1. Cart with a slender rod

A slender rod of length l (2m) and mass m_2 (0.5kg) is attached by a frictionless pivot at 'A' to a block of mass m_1 (1kg). The block moves horizontally on rollers. The position of the block is described by the x coordinate. The block is connected to a fixed wall by a spring of constant k (10 N/m) and unstretched length l_0 (0.5m), and a damper with damping coefficient b (5N-s/m). The position of the block in the inertial frame is $x\hat{i}$.

Use the results from Pset 6, #1 to find the two equations of motion of this system

and linearize them. Use the linearized equations to find the undamped natural frequencies and mode shapes of the system.

Concept question: Do you expect to be able to eliminate the terms involving gravity in the equations of motion by choosing coordinates with respect to static equilibrium positions.

Answer: Gravity will appear in at least one of the EOMs in this problem because it provides the restoring torque on the pendulum. The moment arm is a function of θ and therefore the term involving gravity will involve a coordinate which describes the motion.

Solution: From problem 1 in PSet 6, we have the expressions for potential and kinetic energy of this system. The brute force way to get the equations of motion is to grind through the Lagrange equations. The Lagrange method is very useful for the purpose of having an independent method by which to obtain equations of motion, thus providing a check on equations obtained by application of Newton and Euler's methods. One of the shortcomings of Lagrange is that one learns little about the underlying physics. We shall use both methods here. First, by Lagrange.

Apply Lagrange's equations:

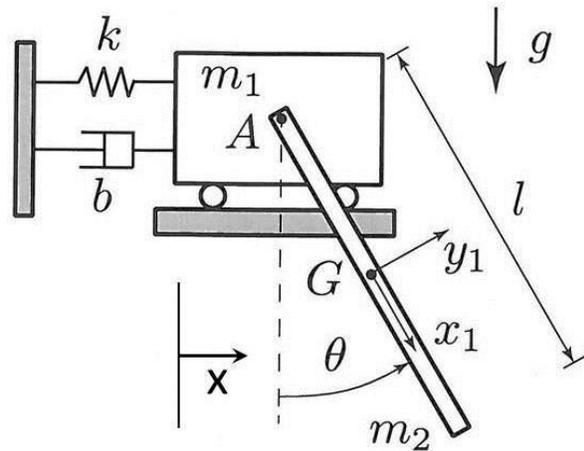
Lagrange equations may be stated as:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j$$
 where $L=T-V$. For purely mechanical systems that have only springs and gravity as potential forces, the Lagrange equations may be more efficiently stated as:

$$\frac{d}{dt} \left(\frac{\partial(T)}{\partial \dot{q}_j} \right) - \frac{\partial(T)}{\partial q_j} + \frac{\partial(V)}{\partial q_j} = Q_j$$
 , where the q_j are the generalized coordinates.

In this particular problem the potential energy expression is given by:

$$V = \frac{1}{2} kx^2 + m_2 g \frac{L}{2} (1 - \cos \theta)$$
 where $x=0$ is at the unstretched spring position, as shown in the figure at right. The coordinate x specifies the translation of the left edge of the cart. Since the cart is a rigid body the translation of every point is the same. The coordinate x is measured from the static equilibrium position for the cart. In the original diagram above $x=0$ was at point of connection of the spring to the wall. If we use the wall as the origin, then we have to deal with the unstretched length of the spring. By repositioning $x=0$ to correspond to the static equilibrium position of the left edge of the cart, this is avoided. The left edge of the cart is also the point of application of a non-conservative force. This definition of the coordinate 'x' also simplifies computation of the generalized forces. x is also an inertial coordinate.



The static equilibrium position for the rod is when it is hanging straight down. The coordinate θ is chosen so that it is zero at static equilibrium. In this way the potential function, V , for this problem is zero at static equilibrium.

The kinetic energy expression was found in PSet 6 to be:

$$T_{rod} = \frac{1}{2} m_2 \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \mathbf{I}_{G/O} \vec{\omega} \quad \text{where } \vec{v}_{G/O} = \vec{v}_{A/O} + \vec{v}_{G/A} = \dot{x} \hat{i} + \dot{\theta} \hat{k} \times \frac{L}{2} \hat{i}_1 = \dot{x} \hat{i} + \frac{L}{2} \dot{\theta} \hat{j}_1$$

$$\frac{1}{2} m_2 \vec{v}_{G/O} \cdot \vec{v}_{G/O} = \frac{1}{2} m_2 \left[\dot{x}^2 + \frac{L^2}{4} \dot{\theta}^2 + L \dot{x} \dot{\theta} \cos \theta \right]$$

$$\frac{1}{2} \vec{\omega} \cdot \mathbf{I}_{G/O} \vec{\omega} = \frac{1}{2} I_{zz/G} \dot{\theta}^2 = m_2 \frac{L^2}{24} \dot{\theta}^2$$

$$T_{rod} = \frac{1}{2} m_2 \left[\dot{x}^2 + \frac{L^2}{3} \dot{\theta}^2 + L \dot{x} \dot{\theta} \cos \theta \right]$$

$$T = T_{cart} + T_{rod} = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 \left[\frac{L^2}{3} \dot{\theta}^2 + L \dot{x} \dot{\theta} \cos \theta \right]$$

Since there are two generalized coordinates in this problem, the Lagrangian must be applied twice. Beginning with the coordinate x the Lagrangian reads as:

$$\frac{d}{dt} \left(\frac{\partial(T)}{\partial \dot{x}} \right) - \frac{\partial(T)}{\partial x} + \frac{\partial(V)}{\partial x} = Q_x, \text{ which leads to}$$

$$(m_1 + m_2) \ddot{x} + kx + m_2 \frac{l}{2} (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = Q_x$$

Finding Q_x , the generalized force is often the most challenging part. It is done here in two ways, the intuitive approach and the vector math approach.

Intuitive approach: The only non-conservative force applied to the system is the dashpot force. If one assumes a positive velocity in the x coordinate, then the dashpot force is given by

$$f_{dashpot} = -b\dot{x} \hat{i}$$

In this problem there are two generalized coordinates, x and θ . If there is a small virtual variation in either coordinate what we need to discover is whether or not it leads to any virtual work being done by non-conservative forces in the problem. In this case a small variation $\delta\theta$ does not result in any movement at the point of application of and in the direction of the dashpot force,

$f_{dashpot} = -b\dot{x} \hat{i}$. From this we draw the conclusion that the generalized force Q_θ must be zero, because there is no virtual work done. Stated mathematically, $Q_\theta \delta\theta = 0$.

Turning to the other generalized coordinate, x , a virtual displacement does result in a movement at the point of application of the non-conservative force. A virtual displacement δx results in virtual work being done as specified by: $Q_x \delta x = -b\dot{x} \delta x \Rightarrow Q_x = -b\dot{x}$.

The vector math approach:

$$Q_x \delta x = \sum \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial x} \delta x, \text{ where } \vec{r}_i = r_i = x \hat{i}, \text{ is the position vector from the origin to}$$

the point of application of the non-conservative force, $-b\dot{x}\hat{i}$.

$$\frac{\partial \vec{r}_1}{\partial x} \delta x = \delta x \hat{i}, \text{ which then allows one to evaluate } Q_x \delta x.$$

$$Q_x \delta x = \sum \vec{F}_1 \cdot \frac{\partial \vec{r}_1}{\partial x} \delta x = \delta x \hat{i} \cdot (-b\dot{x}\hat{i})$$

$$\Rightarrow Q_x = -b\dot{x}$$

Applying Lagrange again for the θ coordinate requires evaluation of the expression:

$$\frac{d}{dt} \left(\frac{\partial(T)}{\partial \dot{\theta}} \right) - \frac{\partial(T)}{\partial \theta} + \frac{\partial(V)}{\partial \theta} = Q_\theta, \text{ which leads to}$$

$$m_2 \ddot{x} \frac{l}{2} \cos \theta + m_2 \frac{l^2}{3} \ddot{\theta} + m_2 g \frac{l}{2} \sin \theta = 0 = Q_\theta$$

$$Q_\theta = 0, \text{ because } \frac{\partial r_1}{\partial \theta} = 0.$$

a. Thus we have two equations of motion:

$$m_2 \ddot{x} \frac{l}{2} \cos \theta + m_2 \frac{l^2}{3} \ddot{\theta} + m_2 g \frac{l}{2} \sin \theta = 0, \text{ and} \quad (1)$$

$$(m_1 + m_2) \ddot{x} + b\dot{x} + kx + m_2 \frac{l}{2} (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = 0 \quad (2)$$

The initial problem statement requires that we linearize these equations and use them to find the undamped natural frequencies and modes shapes of the system. This is done at the end of this solution. Prior to that an alternative way of finding the equations of motion is presented using Newton's and Euler's laws in a very efficient way. The reader may skip this next section and go directly to the solution for the natural frequencies and mode shapes. However, you are strongly urged to at some time read the following alternative derivation of the EOMs, because it is an example of the use Euler's equation in the powerful, but often overlooked form shown here:

$$\sum_i \vec{r}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O}$$

An alternative and very efficient way to find the equations of motion: We shall use a combination of Newton's 2nd and 3rd laws as well as a powerful form of Euler's law, which was presented in the appendix to Problem Set 8. We begin with Euler's law, which in this case may be applied to the pendulum, which rotates about a pivot at 'A'. 'A' is a moving point and

Euler's law for angular momentum about moving points is generally stated as:

$$\sum_i \vec{\tau}_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt} \right)_{O,xyz} + \vec{v}_{A/O} \times \vec{P}_{G/O} \quad (3)$$

The equation above is especially useful when the second term is known to be zero, such as when $\vec{v}_{A/O} = \mathbf{0}$ or when point A coincides with G, the center of mass of the rigid body. When the second term is not zero this equation wastes a great deal of the analysts time, because one must evaluate the 2nd term only to find that it always cancels a term produced in the computation of the first term. The computation of these two terms, which always cancel, may be avoided by using the form below:

$$\sum_i \vec{\tau}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O} \quad (4)$$

'G' is always at the center of mass of the rigid body, which in this case is the rod.

$$\vec{H}_{/G} = I_{zz/G} \dot{\theta} \hat{k} = m_2 \frac{l^2}{12} \dot{\theta} \hat{k}$$

$$\frac{d\vec{H}_{/G}}{dt} = m_2 \frac{l^2}{12} \ddot{\theta} \hat{k}$$

$$\vec{r}_{G/A} = \frac{l}{2} \hat{i}_1$$

$$M = m_2$$

$$\vec{v}_{G/O} = \dot{x} \hat{i} + \frac{l}{2} \dot{\theta} \hat{j}_1$$

$$\vec{a}_{G/O} = \frac{d\vec{v}_{G/O}}{dt} = \ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 + \frac{l}{2} \dot{\theta} \frac{d\hat{j}_1}{dt} = \ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 - \frac{l}{2} \dot{\theta}^2 \hat{i}_1$$

where the relation has been used that $\frac{d\hat{j}_1}{dt} = -\dot{\theta} \hat{i}_1$.

Taking the resulting cross product leads to

$$\vec{r}_{G/A} \times m_2 \vec{a}_{G/O} = \frac{l}{2} \hat{i}_1 \times m_2 \left(\ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 - \frac{l}{2} \dot{\theta}^2 \hat{i}_1 \right) = \left(\frac{l}{2} m_2 \ddot{x} \cos \theta + m_2 \frac{l^2}{4} \ddot{\theta} \right) \hat{k}.$$

To complete the cross product required the substitution of $\hat{i} = \cos \theta \hat{j}_1 + \sin \theta \hat{i}_1$.

Putting it together:

$$\sum_i \vec{r}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O} = m_2 \frac{l^2}{12} \ddot{\theta} \hat{k} + \left(\frac{l}{2} m_2 \ddot{x} \cos \theta + m_2 \frac{l^2}{4} \ddot{\theta} \right) \hat{k} = -m_2 g \frac{l}{2} \sin \theta \hat{k}$$

$$\Rightarrow m_2 \frac{l^2}{3} \ddot{\theta} + \frac{l}{2} m_2 \ddot{x} \cos \theta + m_2 g \frac{l}{2} \sin \theta = 0 \quad (5)$$

This EOM may be checked against equation (2) obtained by Lagrange. To obtain the second equation by an application of Newton's 2nd law, sum the external forces on mass, m_1 . The external forces on the cart are due to the spring, the dashpot and the reaction forces on the pivot at 'A', due to the motion of the swinging rod. Normally these reaction forces would be messy to obtain, but not this time, because they can be expressed in terms of the mass times the acceleration of the rod, which has already been found. In this case an application of Newton's 2nd law to the rod would require:

$$m_2 \vec{a}_{G/O} = \sum_{\text{External}} F_i = m_2 g \hat{j} + (\text{reaction forces at 'A' on the rod}) = m_2 \left(\ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 - \frac{l}{2} \dot{\theta}^2 \hat{i}_1 \right)$$

The above expression may be solved for $R_{x_1} \hat{i}_1 + R_{y_1} \hat{j}_1 =$ "the reaction forces at 'A' on the rod"

$$R_{x_1} \hat{i}_1 + R_{y_1} \hat{j}_1 = m_2 \left(\ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 - \frac{l}{2} \dot{\theta}^2 \hat{i}_1 \right) - m_2 g \hat{j}.$$

Newton's 3rd law requires that the reaction forces at 'A' on the rod, $R_{x_1} \hat{i}_1 + R_{y_1} \hat{j}_1 =$ - the reaction forces on the cart, m_1 .

$$\text{Therefore, the forces on the cart from the rod} = -R_{x_1} \hat{i}_1 - R_{y_1} \hat{j}_1 = - \left[m_2 \left(\ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 - \frac{l}{2} \dot{\theta}^2 \hat{i}_1 \right) - m_2 g \hat{j} \right]$$

The second EOM may be obtained by summing all of the external forces on the cart, m_1 .

$$\sum_{\text{external}} F_i = m_1 \ddot{x} \hat{i} = -kx \hat{i} - b\dot{x} \hat{i} - \left[m_2 \left(\ddot{x} \hat{i} + \frac{l}{2} \ddot{\theta} \hat{j}_1 - \frac{l}{2} \dot{\theta}^2 \hat{i}_1 \right) - m_2 g \hat{j} \right],$$

where the terms inside the square brackets are the forces the rod exerts on the cart.

When the unit vectors \hat{i}_1 and \hat{j}_1 are converted to their equivalents in terms of \hat{i} and \hat{j} , the final form of the EOM is obtained. Since there is no acceleration allowed in the \hat{j} direction, then the \hat{j} terms on the RHS of the equation above must sum to zero, leaving the EOM in the x direction:

$$(m_1 + m_2) \ddot{x} + b\dot{x} + kx + m_2 \frac{l}{2} (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = 0$$

b. The second part of this problem was to linearize the equations of motion.

$$m_2 \ddot{x} \frac{l}{2} \cos \theta + m_2 \frac{l^2}{3} \ddot{\theta} + m_2 g \frac{l}{2} \sin \theta = 0, \text{ and} \quad (1)$$

$$(m_1 + m_2) \ddot{x} + b \dot{x} + kx + m_2 \frac{l}{2} (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = 0 \quad (2)$$

The non-linear terms include the sine and cosine terms. For small rotation of the rod:

$$\sin \theta \approx \theta, \text{ and } \cos \theta \approx 1$$

With this substitution all but one of the non-linear terms disappear.

$$m_2 \ddot{x} \frac{l}{2} + m_2 \frac{l^2}{3} \ddot{\theta} + m_2 g \frac{l}{2} \theta = 0, \text{ and}$$

$$(m_1 + m_2) \ddot{x} + b \dot{x} + kx + m_2 \frac{l}{2} (\ddot{\theta} - \dot{\theta}^2 \theta) = 0$$

The term $\dot{\theta}^2 \theta$ in the second equation is 3rd order in θ , which when θ is small may be neglected, resulting in the following two linearized EOMs.

$$m_2 \ddot{x} \frac{l}{2} + m_2 \frac{l^2}{3} \ddot{\theta} + m_2 g \frac{l}{2} \theta = 0, \text{ and}$$

$$(m_1 + m_2) \ddot{x} + b \dot{x} + kx + m_2 \frac{l}{2} \ddot{\theta} = 0 \quad \text{These may be expressed in matrix form:}$$

$$[M] \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + [C] \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + [K] \begin{Bmatrix} x \\ \theta \end{Bmatrix} = 0, \text{ where } [M], [C] \text{ and } [K] \text{ are known as the}$$

mass, damping and stiffness matrices.

$$\begin{bmatrix} m_2 \frac{l}{2} & m_2 \frac{l^2}{3} \\ (m_1 + m_2) & m_2 \frac{l}{2} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & m_2 g \frac{l}{2} \\ k & 0 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (6)$$

Substitution of the physical parameter values leads to :

$$\begin{bmatrix} 0.5kg - m & 0.667kg - m^2 \\ 1.5kg & 0.5kg - m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 5N - s/m & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 4.905N - m \\ 10N/m & 0 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = 0$$

The units are shown to emphasize that a consistent set must be used. Without showing the units, a very compact statement of the equations of motion results:

$$\begin{bmatrix} 0.5 & 0.667 \\ 1.5 & 0.5 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 4.905 \\ 10 & 0 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = 0. \quad (7)$$

- c. The final step is to obtain the undamped natural frequencies and mode shapes. To obtain a solution we temporarily require the damping and any external excitation terms to be zero. Then assume a solution of the form:

$$\begin{Bmatrix} x(t) \\ \theta(t) \end{Bmatrix} = \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{i\omega t},$$

Taking the second derivative with respect to time provides

$$\begin{Bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{Bmatrix} = -\omega^2 \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{i\omega t}, \text{ which are substituted into the equations of motion:}$$

$$-\omega^2 \begin{bmatrix} m_2 \frac{l}{2} & m_2 \frac{l^2}{3} \\ (m_1 + m_2) & m_2 \frac{l}{2} \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} + \begin{bmatrix} 0 & m_2 g \frac{l}{2} \\ k & 0 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (8)$$

which in generic form

$$\text{appears as: } [-\omega^2[M] + [K]] \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = 0,$$

Element by element the two matrices $-\omega^2[M]$ and $[K]$ may be summed or merged as follows:

$$\begin{bmatrix} -\omega^2 m_2 \frac{l}{2} & -\omega^2 m_2 \frac{l^2}{3} + m_2 g \frac{l}{2} \\ -\omega^2 (m_1 + m_2) + k & -\omega^2 m_2 \frac{l}{2} \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (9)$$

Upon substituting the values for the physical parameters, this equation becomes:

$$\begin{bmatrix} -0.5\omega^2 & -.667\omega^2 + 4.905 \\ -1.5\omega^2 + 10 & -0.5\omega^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

Taking the determinant of equation (9) leads to the characteristic equation, whose roots are the natural frequencies.

$$\omega^4 (m_2 \frac{l}{2})^2 - [-\omega^2 (m_1 + m_2) + k] \left[-\omega^2 m_2 \frac{l^2}{3} + m_2 g \frac{l}{2} \right] = 0$$

$$\omega^4 \left[(m_2 \frac{l}{2})^2 - (m_1 + m_2) m_2 \frac{l^2}{3} \right] + \omega^2 \left[-m_2 (m_1 + m_2) g \frac{l}{2} + k m_2 \frac{l^2}{3} \right] - k m_2 g \frac{l}{2} = 0. \quad (11)$$

This is a 4th order equation in ω , and a 2nd order equation in ω^2 . This is a quadratic equation in the variable ω^2 , which may be solved using the quadratic formula. The squared values of the two natural frequencies that we are looking for are the roots provided by the quadratic formula. To find these roots one must first substitute the numerical values for all of the physical parameters in the problem. Equation 11 then takes on the following form:

$$-0.75\omega^4 + 14.95\omega^2 - 49.05 = 0 \quad (12)$$

The two roots of this equation are the squares of the two natural frequencies of the system. The natural frequencies are found to be:

$$\omega_1 = 2.158 \text{ radians / s}$$

$$\omega_2 = 3.747 \text{ radians / s}$$

Once the natural frequencies are in hand then the mode shapes may be found by inserting, one at a time, one of the two natural frequencies into equation (10), which yields two algebraic equations in the variables X and Θ , as follows.

$$\begin{bmatrix} -0.5\omega^2 & -.667\omega^2 + 4.905 \\ -1.5\omega^2 + 10 & -0.5\omega^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

$$-0.5\omega^2 X - .667\omega^2 \Theta + 4.905\Theta = 0$$

$$-1.5\omega^2 X - .5\omega^2 \Theta + 10X = 0$$

Substitution of $\omega_1^2 = 4.657$ into the first of these equations yields:

$$-2.329X - 3.105\Theta + 4.905\Theta = 0, \text{ which may be solved for } \frac{\Theta}{X} = 1.294$$

Substituting $\omega_1^2 = 4.657$ into the second equation yields:

$$-6.986X + 10X - 2.329\Theta = 0 \Rightarrow 3.0154X = 2.329\Theta = \frac{\Theta}{X} = 1.294$$

One obtains identical results from both equations. They are not independent. At best one can solve for the ratio Θ / X , but not the exact magnitude of either X or θ .

A similar substitution of ω_2^2 into either equation in (10) will yield a second value of $\Theta / X = -1.575$, which is the mode shape corresponding to the vibration of the system at the second natural frequency. For a 2 DOF system there will be two natural frequencies and two values of this ratio, one for each natural frequency.

This problem may also be set up as a numerical problem and solved in a program such as Matlab. For problems with more than two degrees of freedom, a numerical solution is the most practical solution method. In general, all that is required is that one evaluate the mass and stiffness matrices for the linearized equations of motion. These are then used as inputs to a numerical eigenvalue solver. When the physical parameter values are substituted into equations 7 or 8 the following mass and stiffness matrices, $[M]$ and $[K]$ are obtained.

$$[M] = \begin{bmatrix} 0.5kg - m & 0.667kg - m^2 \\ 1.5kg & 0.5kg - m \end{bmatrix}$$

$$[K] = \begin{bmatrix} 0 & 4.905N - m \\ 10N / m & 0 \end{bmatrix}$$

Putting these values into a Matlab program for finding

eigenvalues results in finding the following natural frequencies and mode shapes:

$$\omega_1 = 2.158 \text{ radians/second}$$

$$\omega_2 = 3.746 \text{ radians/second}$$

Mode shapes:

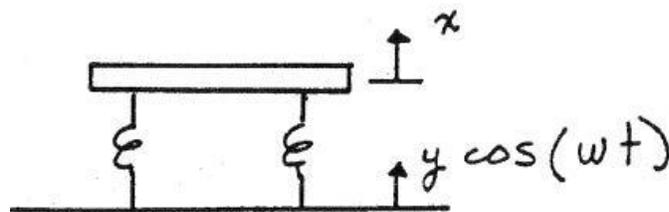
$$\text{Mode 1: } \begin{Bmatrix} X / X \\ \Theta / X \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.295 \end{Bmatrix}$$

$$\text{Mode 2: } \begin{Bmatrix} X / X \\ \Theta / X \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.5749 \end{Bmatrix}$$

For a more detailed description of the solution procedure read the section on solving for natural frequencies in any text on mechanical vibration.

Problem 2

An Optics experiment is set up on a vibration isolation table which may be modeled as a single degree of freedom system for vertical oscillations. The system has very little damping. The floor has considerable vertical vibration at 30 Hz. What must the natural frequency of the table be in order to reduce the motion of the table and optics experiment by 12 dB, when compared to the



motion of the floor? Stated mathematically this would read

$$-12dB = 20 \log_{10} \left(\frac{x}{y} \right).$$

For this part of the problem assume the damping is 0%.

When working on the table, you notice that after an accidental bump the resulting motion takes a long time to die out. To make it die down sooner you add a damper between the table and the floor, such that the total damping is 10% of critical damping. Compute the value of $|x/y|$ with this amount of damping to that from part one.

Concept question: Will the addition of damping increase or reduce the vibration of the table in response to floor motion at 30 Hz?

Answer: The addition of damping increases the response of the table to the floor motion when

the frequency ratio is greater than 1.414. I.e. $\frac{\omega}{\omega_n} > \sqrt{2}$

Solution:

This is an example problem in vibration isolation. The objective is to design a ‘soft’ support for a piece of equipment such that response of the equipment to the motion of the floor is reduced. To do this we need the transfer function for x/y, the ratio of the steady state vibration response, x, to a single frequency steady state motion of the base, y.

The steady state response of a single DOF system to harmonic(single frequency) base motion is governed by the following transfer function.

$$H_{x/y}(\omega) = \left| \frac{x}{y} \right| e^{-i\phi}, \quad \text{where } \left| \frac{x}{y} \right| = \frac{\left[1 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}} .$$

The phase angle ϕ is given by $\phi = \tan^{-1} \frac{2\zeta \left(\frac{\omega}{\omega_n} \right)^3}{1 - \frac{\omega^2}{\omega_n^2} + \left(2\zeta \frac{\omega}{\omega_n} \right)^2} .$

The response x(t) may be written as

$$x(t) = |y| |H_{x/y}(\omega)| \cos(\omega t - \phi)$$

The magnitude and phase angle of this transfer function are shown below. The magnitude of this transfer function has two properties, which are important to the design of vibration isolation systems. The most important is that the ratio of the input operating frequency of the base to the natural frequency of the system must be greater than the square root of 2.

That is, $\frac{\omega}{\omega_n} > \sqrt{2} = 1.414$. This guarantees that $|x/y| < 1$. If the frequency ratio is less than 1.414

the ratio $|x/y|$ will be greater than one and the use of a flexible spring support will make the problem worse than having no soft support at all. The second property is that in the useful

frequency range of $\frac{\omega}{\omega_n} > \sqrt{2} = 1.414$, the greater the damping in the system the poorer the

reduction in the motion. Hence, there is reason to not put too much damping in the design of the system.

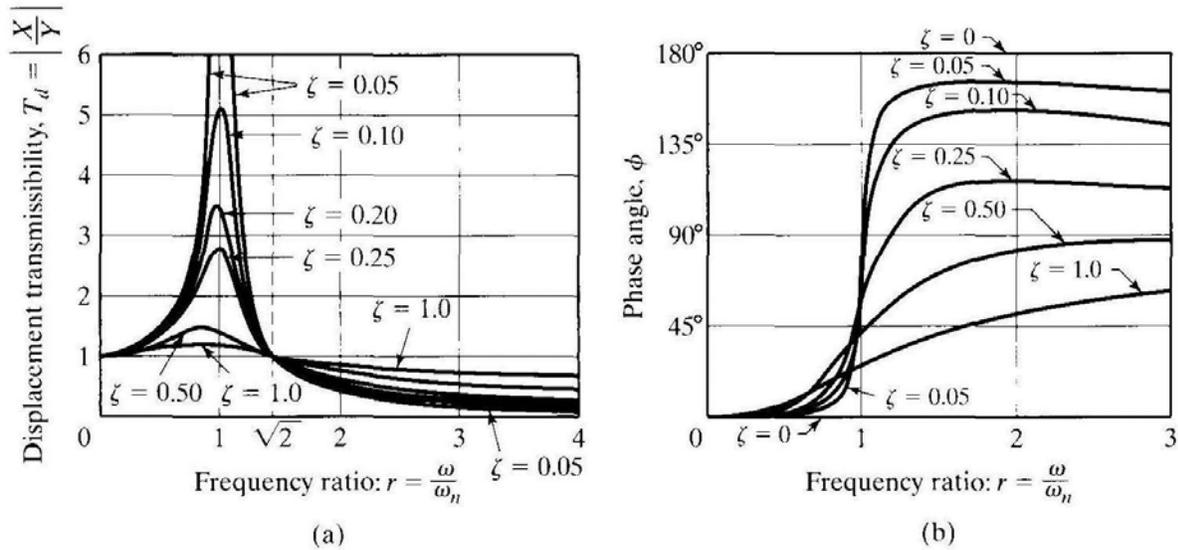


Figure: The magnitude and phase angle of the x/y steady state transfer function.

In the problem statement it is specified that the response should be -12 dB compared to the input. It also suggests that damping is very small. Therefore begin the analysis by setting the damping equal to zero. This will allow you to estimate the frequency ratio that will give the specified reduction in response in the simplest mathematical form. The definition of dB or decibels is shown here as well.

$$\text{decibels} \equiv 20 \log_{10} \left(\left| \frac{x}{y} \right| \right) = -12 \text{ dB} \Rightarrow \log_{10} \left(\left| \frac{x}{y} \right| \right) = -0.6 \Rightarrow \left| \frac{x}{y} \right| = \left| \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \right| = 0.25, \Rightarrow \left| 1 - \frac{\omega^2}{\omega_n^2} \right| = 4.$$

$$\left| 1 - \frac{\omega^2}{\omega_n^2} \right| \text{ can be interpreted as } 1 - \frac{\omega^2}{\omega_n^2} \text{ or as } \frac{\omega^2}{\omega_n^2} - 1.$$

Since $|x/y| < 1$, the only result of interest is that which gives a solution for $\frac{\omega}{\omega_n} > \sqrt{2}$.

$$\text{Thus, use the form } \frac{\omega^2}{\omega_n^2} - 1 = 4$$

$$\Rightarrow \frac{\omega}{\omega_n} = \sqrt{5} = 2.24 = \frac{f}{f_n}. \text{ Since } f \text{ is given as } 30 \text{ Hz, then } f_n = \frac{30 \text{ Hz}}{2.24} = 13.39 \text{ Hz.}$$

$$\omega_n = 2\pi f_n = 84.15 \text{ radians/s} = \sqrt{\frac{k}{M}}$$

At a frequency ratio of 2.24, the response, x, is only 1/4th of the input y: x/y=0.25. This is with

zero damping. The introduction of damping will decrease the performance of the vibration isolation.

b. To find the performance of the system with 10% damping, simply recompute the value of $|x/y|$ with $\zeta = 0.1$.

$$\left| \frac{x}{y} \right| = \frac{\left[1 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}}, \text{ when evaluated at } \frac{\omega}{\omega_n} = 2.24 \text{ and } \zeta = 0.1.$$

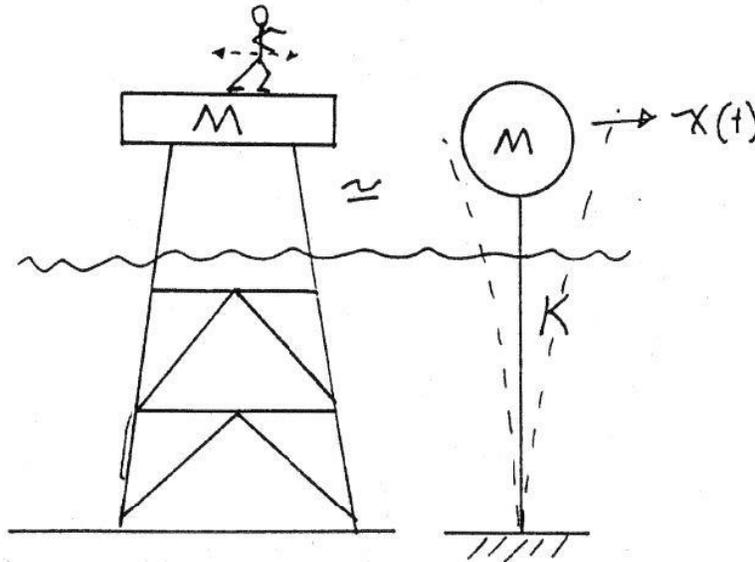
$$\Rightarrow \left| \frac{x}{y} \right| = 0.274.$$

Rather than reducing the response to 25% of the base motion, the introduction of 10% damping results in an increase of the response to 27.4% of the amplitude of the base. However, the introduction of 10% damping dramatically reduces the response of the system to bumps and other transients, which is important for sensitive systems, such as optical benches. Damping also greatly reduces the response of the system to floor motion during the start up phase of whatever is causing the floor motion. The vibration of the floor is usually caused by rotating machinery somewhere nearby, which, when first started up, has to pass through the resonance frequency of the isolation system and will also pass through resonance when coasting down at turn off.

Problem 3

Prof. Vandiver once measured the vibration response of a coast guard light station, which stood in 20 meters of water. It was a steel, space frame structure with a large house sitting on the top. A simple single degree-of-freedom equivalent model of the structure is a massless beam with an equivalent concentrated mass on the end. The equivalent mass is the sum of the mass of the house and 1/4 of the mass of the space frame structure. The total concentrated mass in this case was $M = 2.73 \times 10^5$ Kg. The measured natural frequency of the platform in the first bending mode was 1.0 Hz, and the measured damping was 1.0 % of critical.

By watching the output of an accelerometer on an oscilloscope the professor was able to shift the weight of his body from one foot to the other in synchrony with the motion of the structure at 1.0



Hz. He was able to drive the structure to amplitudes in excess of that observed in a storm with 50 knot winds plus 20 foot ocean waves.

The shifting of his weight from side to side may be modeled as an horizontal harmonic force.

Find the magnitude of the steady state horizontal force that was necessary to drive the structure to a steady state acceleration

amplitude of $0.003 \text{ g}^2 \text{ s}$.

Concept question:

What was the steady state response amplitude expressed in meters: a). 0.003 m , b). $0.003 * 9.81 \text{ m}$, c). $0.003 * 9.81 / (6.28)^2 \text{ m}$.

Answer: The relationship between acceleration and displacement magnitude for steady state harmonic motion is given by

$$|\ddot{x}| = |x| \omega^2, \text{ where } x \text{ is in meters and acceleration is in } \text{m/s}^2.$$

$$|\ddot{x}| = |x| \omega^2 = 0.003(g)9.81 \frac{\text{m/s}^2}{g} = 0.02943 \text{ m/s}^2$$

$$|x| = |\ddot{x}| / \omega^2 = 0.003(g)9.81 \frac{\text{m/s}^2}{g} / (2\pi)^2 (\text{radians/s})^2 = 0.00075 \text{ meters}$$

This is on the order of a millimeter of motion.

Solution:

Harmonic motion is defined as oscillatory motion consisting of a single frequency and with constant amplitude. There are many ways to express this mathematically. Some of the most common are.

$$h(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t) = A \cos(\omega t - \phi) = \text{Re} \left[A e^{i(\omega t - \phi)} \right].$$

When $h(t)$ is a displacement then by taking time derivatives one may find the velocity and acceleration that are associated with the same motion. If 'A' is the amplitude of the motion then the velocity and acceleration are given by:

$$h(t) = A \cos(\omega t - \phi) \Rightarrow \dot{h}(t) = -A\omega \sin(\omega t - \phi) \text{ and } \ddot{h}(t) = -A\omega^2 \cos(\omega t - \phi)$$

It is then easy to remember that the magnitudes of the displacement, velocity and acceleration are: A, $A\omega$, and $A\omega^2$.

In this problem you are given the magnitude of the acceleration and the frequency:

$$\frac{A\omega^2}{g} = 0.003, \text{ where } g = 9.81 \frac{m}{s^2} \text{ and } \omega = (1.0 \frac{\text{cycles}}{s}) 2\pi \frac{\text{radians}}{\text{cycle}} = 6.28 \frac{\text{radians}}{s}$$

Solving for A yields:

$$A = \frac{A\omega^2}{g} \times g / \omega^2 = 0.003 \times 9.81 \frac{m}{s^2} / (6.28 \frac{\text{radians}}{s})^2 = 0.00075 \text{ meters} = 0.75 \text{ mm.}$$

As simple single DOF equivalent model is proposed. It is assumed that by shifting weight a harmonic force is applied to the SDOF system at resonance. At resonance the magnitude of the displacement response of a SDOF system is given by:

$$A = \left| F_o \right| \left| \frac{x}{F} \right| = \frac{F_o}{K} \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}} = \frac{F_o}{K} \frac{1}{2\zeta}, \text{ where } \zeta = 0.01, \text{ and}$$

$$\omega^2 = \omega_n^2 = 6.28^2 = 39.48 (r/s)^2 = \frac{K}{M} \Rightarrow K = M \omega_n^2$$

$$\Rightarrow F_o = A \times 2\zeta \times K = A \times 2\zeta \times M \omega_n^2 = (0.00075m)(0.02)(2.73 \times 10^5 \text{ kg})(39.48 (r/s)^2) = 161.67 N$$

$$F_o = 36.33 \text{ Lbs}$$

36 pounds is a reasonable level of force for a person to exert horizontally on the floor by shifting weight from side to side once per second.

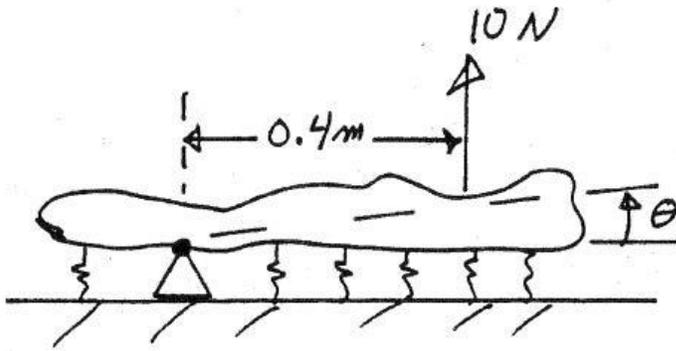
Problem 4

An irregularly shaped object is hinged at one point. It is supported by several springs. Its measured natural frequency is 5.0 Hz. When a static force of 10 N is applied 0.4m from the pivot, the object rotates through an angle of 0.005 radians. What is the mass moment of inertia of the object about the pivot? Assume the body is horizontal when at rest.

Concept question: For small motions about the horizontal do you expect the natural frequency

to be a function of gravity.

Answer: For small oscillations about the horizontal the torque produced by the weight of the object acting on a moment arm $r_{G/A}=0.4m$ is a constant independent of the angle of oscillation. The natural frequency for small oscillations is not a function of 'g'.



Solution:

This is a planar motion problem in which a rigid body rotates about a fixed pivot, designated here as point 'A'. Euler's law takes on a simplified form for such problems given by:

$$\sum_i \vec{\tau}_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt} \right)_{O_{xyz}} + \vec{v}_{A/O} \times \vec{P}_{C/O} = \left(\frac{d\vec{H}_{/A}}{dt} \right)_{O_{xyz}} = I_{zz/A} \ddot{\theta} \hat{k}$$

The sum of the external torques will include that caused by gravity and that due to the springs attached to the object. Assume that the action of the springs may be represented by an equivalent torsional spring constant, $k_{equivalent}$, which will be abbreviated as k_{eq} . It is specified in the problem that this spring constant may be found by an experiment, in which a static torque was applied, resulting in a static rotation of 0.005 radians. Stated mathematically:

$$\vec{r} \times \vec{F} = 0.4m \cos(0.005) \hat{i} \times 10N \hat{j} = 4.0N - m = k_{eq} \theta = 0.005 k_{eq}$$

$$\Rightarrow k_{eq} = (4.0N - m) / .005radian = 800N - m / radian$$

Returning to the application of Eulers law:

$$\sum_i \vec{\tau}_{i/A} = I_{zz/A} \ddot{\theta} \hat{k} \text{ where the sum of the external torques are given by}$$

$$\sum_i \bar{\tau}_{i/A} = -K_{eq} \theta_d \hat{k} - K_{eq} \theta_s \hat{k} - r_{G/A} \cos \theta_d \hat{i} \times (-Mg \hat{j})$$

where θ_s is the static rotation from the zero spring force position to the static equilibrium position. θ_d is the dynamic rotation about the static equilibrium position.

For small θ_d $\cos \theta_d \approx 1.0$ and the sum of the external torques becomes:

$$\sum_i \bar{\tau}_{i/A} = -K_{eq} \theta_d \hat{k} + [-K_{eq} \theta_s \hat{k} - r_{G/A} \hat{i} \times (-Mg \hat{j})] = -K_{eq} \theta_d \hat{k} + [-K_{eq} \theta_s \hat{k} - Mgr_{G/A} \hat{k}]$$

The two terms inside the brackets cancel one another and therefore:

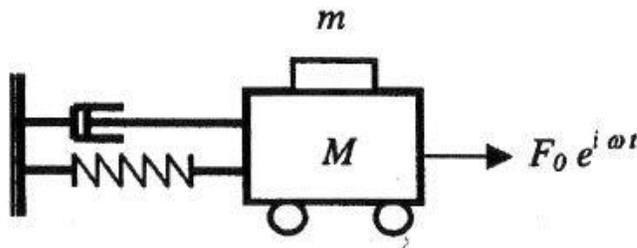
$$\sum_i \bar{\tau}_{i/A} = I_{zz/A} \ddot{\theta}_d \hat{k} = -K_{eq} \theta_d \hat{k} \text{ which yields the final equation of motion:}$$

$$I_{zz/A} \ddot{\theta}_d + K_{eq} \theta_d = 0 \text{ where}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{I_{zz/A}}} = 10\pi \text{ radians/s from which } I_{zz/A} = \frac{k_{eq}}{\omega_n^2} = \frac{800N-m}{100\pi^2} = 0.811kg-m^2.$$

Problem 5

A 1.0 kg mass, m sits on top of a 10.0 kg mass, M . The large mass is connected to a spring, ($K=250N/m$) and a damper, R , and is free to oscillate horizontally on rollers. The damping ratio of the system is 5%. The coefficient of static friction between the large and small block is 0.35.



The large block is driven in steady state vibration by an harmonic force, $F_0(t)$.

- What is the maximum allowable acceleration of the large block, such that the small block does not slide. Express this acceleration in g 's?
- If the harmonic force, $F_0(t)$, is applied to the larger mass at the natural frequency of the system, what is the largest force magnitude, F_0 , that may be applied such that the small mass does not slide on top of the large mass.

Concept question: When the acceleration of the system is one-half that required to make the mass slide, what is the magnitude of the friction force.

- μMg
- $\mu Mg / 2$
- neither

Answer: The magnitude is answer b., one half of the maximum possible friction force. The main point of this question is to shine some light on the frequent misconception that friction force is always μMg . In fact friction force is only as large as it needs to be to get the job done. μMg is an upper bound.

Solution:

- a. The maximum possible friction force is $F_{friction} = \mu mg$. Since the block must obey Newton's 2nd law, then this force is also equal to the mass times the acceleration of the small block. Hence, $F_{friction} = m\ddot{x} = \mu mg \Rightarrow \ddot{x} = \mu g$. Since $\mu = 0.35$, the maximum acceleration in g's is $|\ddot{x}| \leq \mu g = 0.35g$.
- b. This single DOF oscillator is being driven by an harmonic force at the natural frequency of the system. At resonance $\omega = \omega_n$ and the relationship between the input force and the response is governed by the $H_{x/F}$ transfer function, evaluated at resonance.

$$|x| = |F_o| \left| \frac{x}{F} \right| = \frac{F_o}{K} \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(2\zeta \frac{\omega}{\omega_n} \right)^2 \right]^{1/2}} = \frac{F_o}{K} \frac{1}{2\zeta}, \text{ where } \zeta = 0.05, \text{ and}$$

$$\omega = \omega_n = \sqrt{\frac{K}{M+m}} = \sqrt{\frac{250N/m}{11.0kg}} = 4.767 \text{ radians/s}$$

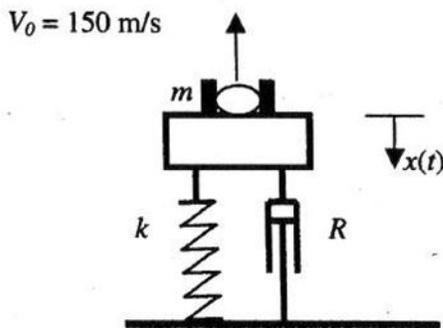
From the solution to problem 3, the relationships between amplitude, velocity and acceleration are known. The magnitude of the acceleration of the total system must not exceed the value found in part 'a'. Therefore the applied force may be computed by:

$$|\ddot{x}| = \omega^2 |x| = \omega_n^2 |x| = \omega_n^2 \frac{F_o}{K} \frac{1}{2\zeta} = \frac{F_o}{(m+M)} \frac{1}{2\zeta} \leq \mu g = 0.35g = 0.35 \times 9.81 \frac{m}{s^2} = 3.43 \frac{m}{s^2}.$$

$$\therefore F_o \leq (m+M) 2\zeta \mu g = 3.77N.$$

Problem 6

A mortar shell is shot almost vertically from a rail car. The rail car is spring supported as shown in the figure. It also has a damper as shown. The mass of the rail car and the mortar is $M =$



$10,000 \text{ kg}$. The mass of the shell is 25 kg . The velocity of the shell is 150 m/s as it leaves the mortar. The spring stiffness of the suspension system of the rail car is $k = 15,791 \text{ N/m}$. The damper constant, R , is 251.33 Ns/m .

- a. Write down the equation of this single degree of freedom system for vertical motion. Determine the natural frequency and damping

ratio (obtain numerical values).

b. The mortar is loaded and fired when the system is initially at rest (no motion). After firing the shell the car will vibrate. Find an expression for $x(t)$, the time history of the vibration of the car, after the shell has been fired. Be specific about the initial conditions that you use. Sketch the result.

c. The vibration of the car will decay with time after firing. Predict the ratio of two vibration amplitude peaks separated by five periods of vibration.

Concept question: What initial conditions will be required? A. initial displacement only, B. Initial velocity only, C. Both initial velocity and displacement.

Answer: C. both initial velocity and displacement will be needed in the answer.

Hint: Use a coordinate system corresponding to the at-rest position of the car after the shell has been fired and the vibration has stopped.

Solution: After the shell leaves the mortar the response of the mortar is describeable in terms of initial conditions only. Due to damping the mortar and rail car will exhibit damped oscillations, which eventually come to rest at a static equilibrium position corresponding to the weight of the mortar plus rail car resting on a spring, but without the mass of the shell. This is the position from which the response coordinate x will be measured. The key is to specify the initial conditions that describe the condition of the mortar and rail car at a very short time after the shell is fired. Let's take them one at a time.

Initial displacement: Prior to firing the mortar and rail car are weighted down with the mass of the shell. This displaces the mortar by an amount

$x_o = mg / k = 25kg \times 9.81 \frac{m}{s^2} / 15,791N / m = 0.016m$. This is downward and positive, because x is defined as positive downward.

Initial velocity: When fired the shell is given momentum

$\Delta \vec{P}_s = m \Delta \vec{V}_s = -25kg * 150(m/s) \hat{i} = -3750(N - m/s) \hat{i}$ From Newton's 3rd law we know that the impulse delivered to the mortar and rail car must be equal and opposite to the impulse given to the shell. Hence,

$$\Delta \vec{P}_M = -\Delta \vec{P}_s = -m \Delta V_s \hat{i} = M \Delta V_M \hat{i} \Rightarrow \Delta V_M = -\frac{m}{M} \Delta V_s = \frac{3750 \hat{i} (N - m/s)}{10000kg} = 0.375 \frac{m}{s}$$

$$\therefore \dot{x}(t=0) = v_o = 0.375 \frac{m}{s}$$

The initial displacement is $x_o=0.016m$ and the initial velocity is $v_o=0.375m/s$. Both are positive downwards.

- a. The equation of motion of this single DOF system is given by $M\ddot{x} + R\dot{x} + kx = 0$, where x is measured from the static equilibrium position of the mortar and rail car without the shell in the mortar.

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{15,791 \text{ N/m}}{10000 \text{ kg}}} = 1.257 \frac{\text{radians}}{\text{s}}$$

$$\zeta = \frac{R}{2\omega_n M} = \frac{251.33 \text{ N-s/m}}{2 \times 1.257 \frac{\text{radians}}{\text{s}} \times 10000 \text{ kg}} = 0.01$$

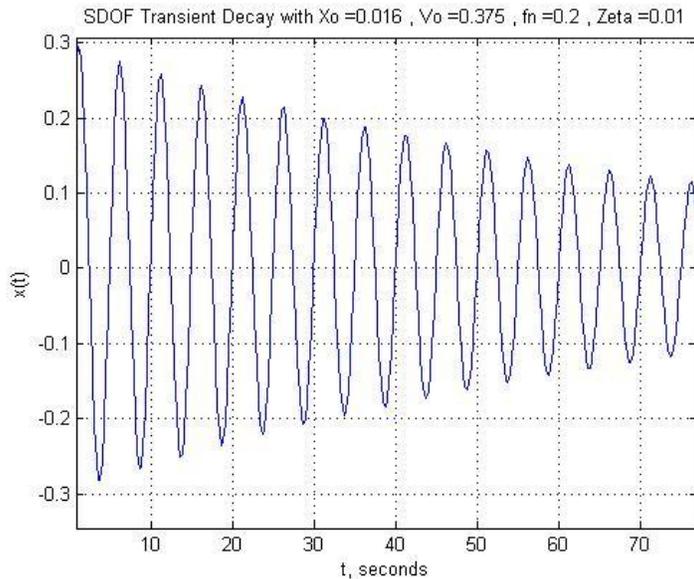
- b. The response to initial conditions for a single DOF system is given by:

$$x(t) = A e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) = \left[x_o \sin(\omega_d t) + \frac{\dot{x}_o + \zeta \omega_n x_o}{\omega_d} \cos(\omega_d t) \right]$$

where

$$A = \left[x_o^2 + \left(\frac{\dot{x}_o + \zeta \omega_n x_o}{\omega_d} \right)^2 \right]^{1/2}, \quad \phi = \tan^{-1} \left[\frac{\dot{x}_o + \zeta \omega_n x_o}{x_o \omega_d} \right] \text{ and } \omega_d = \omega_n \sqrt{1 - \zeta^2}$$

All that is left to do is plug in the numbers and plot the resulting time history.



- c. The problem is to predict the ratio of the amplitude of two vibration peaks separated by five periods of vibration. This may be deduced from the definition of the logarithmic decrement.

$$\zeta = \frac{1}{2\pi n} \ln\left(\frac{A_o}{A_n}\right) = \frac{1}{10\pi} \ln\left(\frac{A_o}{A_n}\right) = 0.01, \text{ and } n=5$$

$$\Rightarrow \frac{A_o}{A_5} = e^{0.314} = 1.369 \text{ or the reciprocal } \frac{A_5}{A_o} = 0.73.$$

After five periods the amplitude of a peak will be reduced to 73% of the initial amplitude.

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