PROFESSOR: So we'll start quickly going over the concept questions for the homework this week. So you have a two rotor system. It says, if you give it a deflection at a exactly the mode shape of mode two, what frequency components do expect the transient response to have? So most people said only omega 2, but quite a few people said both.

So this introduction to vibration that we have been doing for the last few lectures-- my goal in it is to have you folks go away with a pretty good conceptual understanding of the basics of vibration, and on the final exam, I'm not going to have you derive anything, like finding the natural frequencies of a 3 by 3 degree of freedom system, solving a sixth order system in omega squared. I won't do that kind of thing to you, but questions like this are really fair game. So if we did this business bimodal analysis-- and we did an example the other day where with the system we had here, if you deflect it in exactly the shape of one mode, what kind of response do you get? Initial conditions that are shaped exactly like one the mode shapes.

AUDIENCE: It has exactly that natural frequency in it.

PROFESSOR: Well, it'll not only have that natural frequency in it, but if you deflect it initially in the mode shape, what will the responding motion look like?

AUDIENCE: Just the mode shape.

PROFESSOR: Just like the mode shape, and if the motion is in a particular mode shape, it will be at-- this is transient vibration. No external force. You deflect it and let it go. You will see only motion in that mode if you give it an initial deflection exactly in that mode shape. If you went to the other mode, and deflected it that way, and let go, it would change frequency, and it would vibrate only in that shape. And any other combination of motions is some linear sum of those two mode shapes. Any other allowable motion of the system can be made up of some weighted sum of the two mode shapes, right? And that's the amount of each mode that you get-- is the weighted sum.

What are the weightings? If the weightings are 1, 0, then it's all one mode and not of another. If it takes some of each mode to give you the initial deflected shape, that's how much of each mode you get. So the answer to this question is only mode two.

OK, next. I forget what you were given. Apply the concept. Which mode is likely to dominate the steady state response for the excitation of part D? So it was probably being excited harmonically at the natural frequency of mode two. So which mode? So harmonic excitation, which mode do you expect to respond the most when you excite it at one of the natural frequencies?

AUDIENCE: I'd say [INAUDIBLE].

PROFESSOR: At that natural frequency. Why? Why?

AUDIENCE: Because that's what the eigenvalue is.

PROFESSOR: Well, there's an eigenvalue there, but now we're talking about steady state response. What's the transfer function of a single degree of freedom system look like? Just trace it in the air. This is a response per unit input, and where does it go? Way up high--

AUDIENCE: At the resonance.

PROFESSOR: At the resonance. And so if you have two 2 degree of freedom system, how many resonances do you have? What do you think the transfer function is going to look like for a two degree of freedom system if you just plotted it just plot as a function of frequency?

AUDIENCE: Double peak.

PROFESSOR: Double peak. So the transfer function is likely to do that at one frequency and that at the next natural frequency. And that's what we're going to talk about today, in fact.

OK so if you excited the second natural frequency and it's lightly damped, it's likely to be dominated by that modal response. In that question, is it guaranteed to be only that mode responding, even if you derive it at the natural frequency? Another way to saying this question, can the first mode have some response at any frequency you excite the system at? Sure. It has transfer function. It's continuous in frequency. It just won't be very big.

OK, next. For what value is omega over omega n is the force transmitted to the wall going to be the greatest. So this is-- when we make this, this is now we're doing-- it's rotating around. OK, so this thing is rotating mass now at a constant frequency. So it's got a static imbalance.

And at what frequency would you expect the force transmitted to the wall be the greatest?

AUDIENCE: At resonance.

PROFESSOR: At resonance, because that's when you get the most--

AUDIENCE: The most response.

PROFESSOR: Greatest response. And the force transmitted to the wall is through the spring and through the dash pot, and the bigger the motions, the bigger those forces are going to be. OK, next. And so this is-- now, if you'll account for the spring, do you expect the counting for the mass of the spring to increase the predicted natural frequency or decrease it?

AUDIENCE: Decrease.

PROFESSOR: Decrease. Natural frequencies behave like square root of k/m kind of thing. And if you do anything that drives up the m, frequency is going to go down. Another way of asking the same question is, if you neglect mass when you're estimating the natural frequency of a system, which way are you likely to be in error? This system, normally we just ignore the mass of the springs, and we calculate the natural frequency as square root of k/m, and we're almost certain to be-- predict what? Too high or too low?

AUDIENCE: Too high.

PROFESSOR: Too high, because we've ignored some mass in the system. OK, next. I think that's it. I want to get on with today's lecture. We have done vibration from point of view of single degree of freedom systems, transient response, and steady escape response. And then we've started looking at multiple degree of freedom systems, but we actually started it from the point of view of modal analysis, and we looked at two degree of freedom systems, and found the modal contributions of each of the modes, and sort of did them one at a time.

So is there another way-- one of you asked me after class last time, well, can't you just solve the differential equations directly and not bother with separating the modes out and figuring out the modal contributions. The answer is yes, and you do it via transfer functions, except now you will need more than one if you have more than one degree of freedom. So that's what we'll focus on today.

And we're going to do it by thinking about the example actually from the homework. So in the homework you have this problem of this cart with this pendulum hanging off of it. And we-- in problem three, I think it is-- is say, OK, now the pendulum is going to go around a constant rate, and it's going to turn this into a rotating imbalance problem. And I shorten it up so it's just a little mass spinning around. You have k1, c1, x1 here.

And I think you have draw in there as theta. This is point A about which it spins. And you know the non-linear equation to motion for treating this as a two degree of freedom system, just pendulum on a cart. And then nonlinear equation of motion.

And I'm going to put subscripts here, and I'm going to call this coordinate x1, because we're going to need a second coordinate-- or actually, do we? What am I saying? We don't need to do that. Just x will do. We do need the m1 and m2, though. And so that is all-- I'm going to move the remaining stuff to the right hand side. So this is your nonlinear m2L theta double dot cos theta minus theta dot squared sine theta on the right hand side. And I think I need a minus if I do that.

Now, what I do in this problem, I say let's let theta dot equals omega, and that's a constant. And so that says theta double dot is 0. When you do that, then this term goes away, and this becomes omega squared minus times a minus. This equals 1/2 m2L omega squared sine. And theta is just omega t.

So by forcing what was previously an unknown variable that you had the solve in this-- you had two equations and two unknowns because you had two generalized coordinates it took to describe the motion of this thing, x and theta. Now, once you prescribe theta and it's given, it's no longer a variable. It's no longer a coordinate in your generalized-- in your equations of motion that you have to solve for, and it reduces this equation to the equation of a single degree of freedom system. On the left hand side is the response quantities just like usual-- mx double dot plus bx dot plus kx. And on the right hand side is our harmonic excitation, and that's this unbalanced mass going around and around.

So this has the form here. This is some-- if you want to think of it this way, this is some F0, and it looks like sine omega t. So this is just a single degree of freedom system excited by harmonic excitation, and you know it has a transfer function, and you know that there's going to be a resonance at the natural frequency of the system. And if you don't like working with sine omega t, you could say, well, let's measure omega theta from here if you want to, and now that's cosine omega t.

OK, so that's sort of the set up. Let me say one-- if I asked you to solve for the magnitude of the response x1 here, how would you do it? So this is now a single degree of freedom system excited by harmonic excitation. Steady state response-- how do you do it? This you do have to know.

AUDIENCE: Transfer function.

PROFESSOR: Use a transfer function. So this going to be the magnitude of the force times the magnitude of the HxF of omega transfer function, and that's how you get the magnitude of the response, and it happens to be a resonance. Then you'd be at the peak, and if you're not a resonance, wherever you happen to be in frequency is where you would evaluate this, and there's your response.

Now, a few years ago-- every year-- twice a year, in fact-- and they're coming up in January, we have doctoral exams for students who want to do PhDs in mechanical engineering, and most departments at MIT have these also. And in the dynamics portion, there's an oral exam part. There's a written exam, also, but in the oral exam in this particular year, gave a single degree of freedom system.

And we posed-- and we all know that, if we excite this thing with a harmonic excitation, some F1 cosine-- and I will use complex notation, because we're going to need it in a minute-- some F1 e to the i omega t. That's the excitation. We know what the steady state response of this looks like. The magnitude looks like that transfer function. But we posed-- so most would be doctoral students would know all about single degree of freedom [INAUDIBLE].

So we posed the following question. Well, we know that the response of this is going to do something like that, and you evaluate this at whatever frequency you're interested in, including right at resonance. And the question we asked is, is it possible to add a second spring and a second mass, m2 and k2, such that-- so is it possible to pick a k2 and an m2, such that, with this excitation on here, the motion of that thing will be 0? And in order to solve this problem, as soon as you put this on here and assume it's sliding along-- it can't fall. It only has horizontal possible motion, and we'd give it some coordinate describing its motion x2.

So now how many degrees of freedom does this problem have? Two. How many equations of motion do you expect? How many peaks in a transfer function would you expect to see? Two resonances.

So now we need to know how to find transfer functions for a multiple degree of freedom system. enough in everything I've said about two degree of freedom systems, everything is generalizable to n degrees of freedom. Though, what I'm going to show you now is how to do transfer functions for multi-degree of freedom systems, and I'm going to do it by way of example of a two a degree of freedom one, but just keep in mind that you can completely generalize this.

So there is my system-- two masses, two springs-- and I could even have dash spots in here-- a c1, and I might have a c2. And I could even, in general, additionally have a force acting on this second mass, which I'll call F2e to the I omega t. So that's the completely general problem now.

Now, the equations of motion for this system you could write down. We've done it many times now-- this particular system even-- but we could write them in matrix form as mx double dot plus cx dot plus k. Stiffness matrix x equals and excite the excitation in this two degree-- well, I'll just keep this is completely general. This is any n degree of freedom system, some Fe to the i omega t.

So here's kind of an important point. We solve for the motion of a multiple degree of freedom system to a harmonic input. We do it one frequency at a time. So this doesn't mean that I could have force on the first mass at omega 1 or at some omega, and force of the second mass at some other omega. I'd never do that at the same time. It's a linear system. Superposition holds. You do things one frequency at a time, get the answer, and if you have another frequency part, you do that separately, then add the two answers. So this is assumed. All the forces acting on the masses are assumed to occur at one frequency, but you can't have different amplitudes on the different masses. So that's what that means.

So this then for the two degree of freedom system-- F, for examplet-- F of t would be some constant F1. Another constant magnitude, F2e to the i omega t. That's what we mean by that.

Actually, if you do that, if these are just constants, some constant force vector applied at a single frequency, what frequency do you expect to see in the response of the system? Steady state response.

AUDIENCE: Driving frequency. Driving frequency.

PROFESSOR: Right, it's a linear system more or less since you really want to go away with. This is an intralinear system. Steady state response of a linear system-- the frequency in is the frequency out. Always true. So we expect to see a solution. We're going to get a solution of the form x equals some magnitude vector times an e to the i omega t also. And the magnitude might be complex, because it there be phase angles there due to damping and so forth, but nonetheless, they're constants. This part is a constant vector, and that's its frequency dependence.

And if we know that is true, then we can say, well, take the time derivative of that. The only time dependent part is the e to i omega t. So x dot becomes i omega xe to the i omega t. And x double dot becomes minus omega squared xe to the i omega t. So we can substitute this, this, and this into this equation. Of course, these are matrix equations.

And when you do that, minus omega squared times the mass matrix i omega times the c matrix plus k xe to the i omega t equals Fe to the i omega t. You can get rid of these, and now you have a algebraic equation that's no longer a function of time-- function of your original mass, damping, and stiffness matrices. And it's certainly a function of frequency.

And if you think back, this is how we derive the transfer function for a single degree of freedom system. And this statement is true for single degree of freedom, too. With single degree of freedom, that's just the mass. That's just the damping, and that's just the stiffness. And you could solve directly for the hx/f transfer function, but with multiple degrees of freedom, we can't just quite divide this out.

This piece here is known as the-- I'll write it is z omega-- z of omega. This is known as the impedance matrix. And if I want to solve, I'm looking for x. I want to know my solution x. So this is essentially of the form z. It's a matrix times x, a vector, equals F, a vector.

And just using what you know about linear algebra, to solve for x, you just multiply through by z inverse. So x equals z inverse F. All right? And z is a two degree of freedom system. z is a two by two matrix with frequency and everything in it, but it's a two by two. So z inverse will be a two by two.

And for two by two, we can just write down the answer, but let's see. I'll write out z of omega here. So just to be clear about what all this is, z of omega is this, and in this problem, that's minus omega squared times a mass matrix, which looks like m1, 0, 0, m2. And you add to that a damping matrix, i omega, times c. The c is from the equations of motion. c1 plus c2, minus c2, minus c2, c2, and plus our k matrix. k1 plus k2, minus k2, minus k2 k2. So that's what the z of omega actually looks like.

And so you would collect-- so it's a two by two matrix, and its upper left term is minus omega squared m1 plus i omega times this plus that. Collect them all together. And this has a form-- this collects together in a form we call z11, z12, z21, z22. And very often z12 and z21 are symmetric. Not always. The kind of problems we generally do here, they will be, and it will be generally symmetric if your coordinate system is measured from a static equilibrium position.

So if you've been doing 2001, there's a thing called Maxwell's reciprocal theorem, which proves this for the stiffness matrix. They essentially need to be measured from static equilibrium positions, but for today's example, this is indeed symmetric. Minus c2, minus c2, minus k2, minus k2. 0, 0. Everything is symmetric.

So we'll take advantage of that, and I'll write out one of the components here-- z11, for example, when you collect the terms together, is in general, it would be minus omega squared m11 plus i omega c11 plus k11, the corner elements of these three pieces. And in this problem, that's minus omega squared m1 plus i omega c1 plus c2 plus k1 plus k2. You just substitute in this, this, and this. So that's z11, for example, and the other ones you can figure out what each of the other terms would be.

All right, so we've said that we want to know what the x's are. And we know that we can get that by doing z inverse times F. And this gives us our definition. This z inverse is our H, our transfer function matrix, times F. So just by getting z inverse, we get this little set of four transfer functions that we're interested in.

So H is an N by N matrix of transfer functions. So in a two by two-- or for the two by two case, our N equals 2. We know we can write out directly what z inverse is. And z inverse-- z22, minus z12, minus z12 when it's symmetric, z11, all over-- how should I write it? All over the determinant of z. So this is the determinant. So the determinant of z [INAUDIBLE] do these double bars. So this over the determinant is the inverse of that matrix, and this gives you a new result, which is H11, H12, H12, H22, where Hij-- this is the response at i per unit force at j.

And I've made this-- I took advantage of the symmetry here, but this one would normally be called 21, so this is response at 1 caused by the force at 1. So we have forces in our problem. One is the force on the main mass. So how much response do you get at the main first mass due to the force on the first mass? This is how much response you get on the first mass due to the excitation in the second mass. Response on the second mass due to the excitation on the first. Response in the second mass due to the excitation on the second mass. So you get four possible contributions here. And the determinant of z for a two by two is also very straightforward. z11, z22, minus z12 squared.

So we can and will work out exactly what the algebra tells us for this two by two case. So we've already been discussing it, but what do you expect. Let's say let's look at this one. What do you suppose the response at 1 due to a harmonic excitation at 1-- what do you think the sketch of the magnitude of H11 of omega would look like as a function a frequency?

AUDIENCE: Will C12 and C21 always be symmetric?

PROFESSOR: No, but for the kind of problems we do, probably. And if it's simple beams, or masses connected by springs, if you choose your generalized coordinates in a way that is, for example, measuring the displacement of each mass from an inertial static equilibrium starting point, it'll be symmetric. But if you measure the-- even in this two degree of freedom, I can make it non-symmetric just by choosing the coordinate for the second mass as to be relative to the first mass. Soon as you do that, it makes it non-symmetric.

You can still solve for transfer function and everything, but it gets messier. Though, notice I picked coordinates-- x1 relative to the inertial frame, x2 relative to the same inertial frame-- and they're both measuring displacement from a static equilibrium position. Then they are symmetric. And there's the determinant we need, and this determinant-- see, this is divided into all four of these terms. So these four transfer functions all share the same denominator. They all have the same denominator.

So I just want you to use your intuition. Tell me what this is going to look like. What's it look like for a single degree of freedom system? It's HxF for a single degree of freedom-- it's H11 for a single degree of freedom system as a response of those systems to a harmonic force on that single mass, and that looks like what? Peak, right? This one-- what do you think it's going to look like? Show me again. OK, but I see one peak. How many peaks?

AUDIENCE: Two.

PROFESSOR: Why?

AUDIENCE: Because it's two degrees of freedom.

PROFESSOR: Two different natural frequencies. This thing is going to look something like this, and I don't know quite how it behaves in here, but it's going to do that for sure. It's going to have two resonances-- this one at omega 1, this one at omega 2. We already know that because we did it by modal analysis. We know each mode is going to have a resonance in it.

OK, so let's find out exactly what it looks like. So for this two degree of freedom system, there's an F1 and an F2, and we're looking for x1. So x1 x2 here equals H11, H12, H12, H22 times F1 F2. So if I want to solve for x1, H11 F1 plus H12 F2. And for the problem I'm going to solve today, I'm going to let F2 be 0, just to keep it simple, but also to address the original question we ask the doctoral students.

You have a force on the first mass. Can I add a mass and spring to it in such a way that I can make the response of the first mass go to 0. So there is no second force. There's only a first force, and so this term goes away. And so this problem, x1, is H11 F1. Therefore, we need to know what H11 looks like.

So H11 is going to be z22 divided by the determinant. So x1 is going to be z22 over Z11, Z22 minus Z12 squared times F1. And actually I can do this so-- and I'm going to let, for now, damping be 0, because it also simplifies it for the purposes of illustration today. And that's how I was posed to the doctoral student. Didn't even show any damping. It was just a simple system, a string and a mass. How can you do this? So we'll let the damping be 0 and that considerably simplifies things then.

So Z11 is minus omega squared m1 plus k1 plus k2. z22 minus omega squared m2 plus k2. And z12 just minus k2. So now these are pretty simple expressions, and all I have to do is plug them in here. And notice what's going to happen. z11 times z22-- that's something involving omega squared times another term involving omega squared. It's going to give you a polynomial and omega to the fourth. This is the polynomial that has two roots, and the two routes are?

AUDIENCE: [INAUDIBLE] frequencies.

PROFESSOR: Yeah, natural frequencies. The determinant of the z matrix is the same problem you solve when you've found the roots of the characteristic equation. It is the characteristic equation for the system, the denominator, the determinant of the z matrix. Therefore it's going to be a fourth order equation in omega. It'll give you two routes for omega squared. They're your two undamped natural frequencies when we leave damping out of this expression, and they'd be damped-- the damped expressions-- they'd have complex stuff if you leave in the damping. But they give us our two roots, and they go in the denominator. Yeah?

AUDIENCE: If we set the denominator or the determinant of the matrix equal to 0 and solve, you get the modal frequencies?

PROFESSOR: Yeah. That denominator is the characteristic equation that we did when we set it up to do the natural frequencies. Same equation. So we know that this thing has two roots, one at each of the natural frequencies. And when omega is at a natural frequency, what is the value of this characteristic equation? Well, no. The characteristic equation. Think of the denominator. Add a root, that expression goes to? 0. So you're dividing by 0. That's what causes the peaks. That's where the peaks come from. That's where the denominator goes to 0, and that's at the natural frequencies.

OK, so x1 for this problem equals H11F1, and now we can just write it out. H11 is minus omega squared m2 plus k2 times F1. And the denominator, the characteristic equation, is minus omega squared m1 plus k1 plus k2 times minus omega squared m2 plus k2 minus k2 squared. So this whole thing down here-- there's your characteristic equation. You multiply it out. You get your fourth order polynomial. Solve it. You get two roots, but this is now a total expression for the response we were looking for.

So right away this gives me the answer to the question that was posed to the doctoral students. So is there a value-- can you set k2 and m2 such that you can make the response of the system 0 at a particular omega? All you have to do is make the numerator go to 0, right?

So if you make this go to 0, x goes to 0. So x1 equals 0 when minus omega squared m2 plus k2 equals zero, and that happens when k2 over m2 equals omega squared. So how many-- once you set k2 and m2, how many frequencies does this happen at? At how many different operating points can you make the response of that main mass go to 0? Well, just one. Once you choose k1 and m2, whatever you've chosen to be, they give you some value of omega. And when I said I was kind of vague about what happens here, this is H11. Looks like that. Right here is when omega squared equals k2/m2.

AUDIENCE: So when it's at the frequency that makes the numerator 0, wouldn't it also-- I don't remember. Would the denominator be 0 also?

PROFESSOR: No. No. Because, look at the picture here. Where are the two natural frequencies of this system? There's one here, and there's one here. And that's when the denominator goes to 0. When the denominator goes to 0, the response goes to infinity with no damping. So the system has two natural frequencies-- one to either side of this point.

So I've picked kind of a particular example to illustrate the use of a transfer function, and the fact that you can have transfer functions for multiple degree of freedom systems, and they essentially become transfer function matrices. And you use what you need, and for this problem, we needed H11. We worked it out. There it is. This is a complete equation that describes, for me, the behavior of mass 1 per unit input force as a function of frequency. And it will always have a point right there that it goes to 0.

And we don't have any damping in here, so intuitively, what do you think damping will do to this plot? What's the first thing it does to the peaks? What? They become?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Finite. And so the small damping, you're going to have very high tops. More damping, they're going to be lower. So damping especially affects the peaks. Damping will also pull this off the bottom a little bit. It won't perfectly go to 0.

So I want to do a very particular case. So an obvious one is your original system. It might be just a mass spring with a rotor in it that's unbalanced, and it's running near resonance, and it's vibrating like crazy. A single degree of freedom system vibrating like crazy. Can I put on a second mass spring and stop the motion? Well, we can theoretically.

And we'll just say let's let the problem we're trying to solve-- that first system had a natural frequency. I'll call it cap omega n that was the square root k1 over m1, and I'm going to let that be equal to square root of k2 over m2. So I've now chosen k2 and m2 such they're exactly at the natural frequency of the original single degree of freedom system. So this is the original system, k1 m1, and it has that natural frequency. And now I'm going to stick on a second mass, k2 m2, but the value k2/m2 of the second little system-- by itself, it has the same natural frequency as the first if you want to think of it that way. If I make this fixed, what's the natural frequency of that mass and spring? Same natural frequency. I'm just making it that way.

So this is now my system. That's the parameters, and now we can-- I'm going to give you a plot of the response of that particular system. You know how we non-dimensionalize the single degree of freedom transfer function? We set x divided by the static motion. I want to do that again here.

So let's look at this system. It's a two degree of freedom system. It has a force F1 on it, and as the frequency goes to 0 and has displacement x1 and x2-- so at omega equals 0, x1 equals x1s. I'll call it x1 static. And how much is it? So what's the static displacement at 0 frequency of this system to that force? Come on. It's just a spring and a force. How much will that spring stretch when you put a force F1 on it?

AUDIENCE: [INAUDIBLE].

PROFESSOR: F1/k. So x1 static equals F1/k1. And F2 is 0. There's no F2 force. So how much is x2 static? There's no forces on the second mass. How much will it move?

AUDIENCE: [INAUDIBLE].

PROFESSOR: It just moves the same amount as a first mass. So this is x2 static, because the forces on the system are F1 and 0. There's no force in this. It's not going to do anything to that spring. It'll just move with the whole system.

So I want to plot x1 over x1 static. In general, x1 is a function of frequency-- is F1 times-- and I'm going to do the magnitudes here-- times the magnitude of H11. And I'm going to divide that by F1 over k1. So the F1's go away. k1 comes in the numerator. This looks like k1 h11. So a plot of k1 magnitude h11. We know roughly what it's going to look like. It has a peak. It has a 0. It has another peak.

Over here, this plot goes to 1. There's x1 over-- this is also x1 over x1 static. Same thing. It goes to 1. Here it goes to 0. Here is omega 1. Here is omega 2. And right here you're at the original cap omega n, because that's the way we've designed this thing, and we know this is the way H11 behaves. It goes to 0 at the value we've chosen for square root of k2/m2.

So we started out with an original system that had a natural frequency here. k1/m1 was right here. We stuck on this second little mass, and it made into a two degree of freedom system. It no longer has a natural frequency there. It has a natural frequency below it, and a natural frequency above it, and a 0 right where the original natural frequency was.

Imagine what's going on in the system. You've got this force being applied to the first mass, and the first mass isn't moving. Do you think the second mass is moving? Think about the free body diagram of the first mass. F equals ma. There is a force on that first mass-- F1 cosine omega t-- and it's not moving. Mx1 double dot is 0. So in order for the thing not to move, there's a force on it. There must be some other force exactly canceling it. Where does it come from?

AUDIENCE: Mass two.

PROFESSOR: Mass two through the spring. OK, so I'd also like to know, what is x2? Well, x2 from our transfer function matrix is the response of 2 due to a force at 1 times F1, plus a response at 2 due to a force at 2 times F2, but that second term is 0 because F2 is 0. That's a one term expression, and H21 is z11 over the denominator. And Z11-- so x2 is going to look like that.

I wrote that wrong. This is not z11. It is z minus z12 like that. And if you work that out, it's F1 times k2 over the same denominator, but let's evaluate-- here's our-- this is the frequency we've been interested in. Let's evaluate the response of x2 right at this operating point. So let's evaluate this at omega equals cap omega n.

So that's k2 down here minus omega squared m1 plus k1 plus k2. It's the same denominator as always. k2-- I'm not going to write it out. It's exactly the same thing as before, but I want to plug in to this denominator the operating frequency. We're going to operate at cap omega n, which is k1/m1 squared-- square root of k1/m1, or square root of k2/me. I'm going to plug in in this, and let's see what we get.

And I'm going to let omega equal k1/m1 or k2/m2. It's the same thing, and I'm plug them in whatever is convenient. So here I'm going to put k1/m1, N1 and the m1's are going to cancel. I'll be left an k1. F1 over k1 and down in the denominator, this term turns into k1/m1, so that's minus k1 plus k1 plus k1. Those two cancel. I just get a k2. Over here, I'll use minus k2/m2 times m2 plus k2, and that gives me a minus k2 plus k2. This whole thing goes to 0, and I'm left with minus k2 squared and in the denominator a k2. This whole thing turns into F1 over k2.

So the total response of that second mass at this operating frequency is just F1 over k2. And I would like to plot it as what does x2 over x1 static-- and x1 static and x2 static are the same. Well, that is F1 over k2 over F1 over k1. The F1's go away, and I get k1/k2. That also happens to be m1/m2, and it's a quantity we call 1 over mu. And we have just designed what's called a dynamic absorber.

And a dynamic absorber is a little device you can use to stop vibration. So when we were talking about vibration isolation and vibration mitigation a few days ago and I said you've got some rotating imbalance [INAUDIBLE] or something to shake, well, give me three ways of solving it. We said balance the rotor. What were the other two ways of perhaps reducing the vibration?

AUDIENCE: Adding a [INAUDIBLE] damper?

PROFESSOR: Yeah, but also if you've got something that's shaking like crazy, and it's putting fibrillation into the floor or into the table, you can isolate it with a mass and a spring, or a microscope over here that's vibrating like crazy, you can isolate it with a mass or a spring. So this is to stop a vibration isolation, which is guaranteed to be on the final-- the simple practical applications of single degree of freedom stuff.

So we have three ways-- fix the rotor, isolate it with a mass and a spring, isolate the sensitive instrument mass and a spring. Now you have a fourth way. If it's a particular operating frequency, we can operate right here with this thing called the dynamic absorber. This mu quantity is called the mass ratio. That's m2/m1. Basically, these things are real. They're actually used in real machines, and usually you can't-- this dynamic absorber thing you stick on there can't be as big as the original system. It's going to be some small fraction of the original system size-- 5% or 10% if you're lucky. So the bigger this thing is, you'll find out the better it works.

So how much is this little second mass bouncing around? Well it's bouncing around compared to x1 static in the ratio of k1 to k2, which is this 1 over mu quantity. So if mu is 10%-- if you've added the second mass, it's 10% the size of the first one. 1 over mu is a factor of 10. So this transfer function basically looks-- do I have some colored-- so the x2/x1 transfer function basically behaves the same right here and pretty much the same way out here. And in here, it comes down like this and comes back up. And this height right here is 1 over mu.

So the smaller you make this thing, the tinier you make it, the more it has to shake to force the first mas to go to 0. So basically, the reason this thing works-- the free body diagram-- at the 0 point, the main mass m1, x1 equals 0 here. It's not moving. It's got a force acting on it that is some F1 cosine omega t. It's got a spring force acting on it from this second mass going. Here is m2 over here, and it's going back and forth like crazy putting force through this spring. And the spring force had better be exactly equal to that.

So the F spring is going to be equal to minus F1, and that will be equal to x2 times kx2. The second mass has to move enough that it'll compress the spring enough that it provides a force equal and opposite to this one so that it's in equilibrium and it doesn't move. All right. Yeah?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah.

AUDIENCE: So the building, if you want the masses to be fairly equal to each other, how is that ever going to happen?

PROFESSOR: It's not. Not usually. Dynamic absorbers are used in real things, like the Hancock Building. And they're used in engines. They're used in all sorts of little devices that you're not aware they they're there. And you have to usually hide them inside the footprint of the original device. So you don't want them-- for a real thing, you don't want to have a pump with this huge appendage on it. It's just not practical. So they tend to be small.

And so, as a consequence, in order to make them work at this particular frequency, you have to vibrate like crazy. So the design cost of one of these things is you have to design to be able to allow this second mass to shake like all get out. Now, how many frequencies does this work at? Once you've designed it, it only works at one frequency, so this is only useful for something that has a fixed operating frequency, like a synchronous motor. This isn't a good thing for something that has a range of frequencies over which it can work, but it is a different, more complicated theory. There's a type of dynamic absorber that you can make work over a wide range of frequencies.

I need to tell you one other thing. When we put this thing on, the original natural frequency of the first system was here, and it created two new ones. You had a further question.

AUDIENCE: So how does making it vibrate like that help? It's only going to work at that one frequency, and you're always going to have that gap. So making it vibrate really fast, how does that help at all?

PROFESSOR: If, for some reason, you really don't want that first thing to shake, this makes it stop shaking at that frequency. So for example, one of the widest distributed textbooks in the world was written by an MIT professor named Den Hartog, and it's called *Mechanical Vibration*. It was written in the 1930s. It's got a wonderful chapter on dynamic absorbers in it. And the example that he gives of a real device that actually uses one was an electric hair clipper that a barber uses. The head on a hair clipper-- it has to go back and forth like this in or to cut the hair. That is an oscillating mass, and you're going to feel it. You're going to feel the mass times the acceleration-- ma omega squared acceleration-- at the frequency it runs that.

So they did a clever thing. They built inside the case of this thing a little second mass and spring. And so that when you're holding the clippers, it doesn't feel like it's-- OK? So that was an example. It was actually a great example in his 1938 textbook.

And just so you know, you really do have two natural frequencies when you do this, and I'll write them like this. Omega 1 squared and omega 2 squared are the original frequency squared times 1 plus mu over 2 plus or minus mu plus mu squared over 4 square root. So as your mass ratio gets bigger, the two frequencies move further and further apart. When the mass ratio is 0, this quantity-- they have two natural frequencies. It goes to 1 natural frequency. It's a single degree of freedom system, and it's at the original value. As the mass ratio gets bigger and bigger, these two roots-- this one and this one-- as you add mass to that second system, they spread apart. And the bigger the mass ratio, the further apart the two natural frequencies become.

So we've got 10 minutes left, and I have a demo which illustrates this thing. It's a little delicate to make it work, because actually it was really frustrating. I had it all set up. It worked great in my office. Came in 10 minutes early today to set it up, and the table in here is so flexible I can't make it behave like a fixed surface. So I tweaked it, and I think I've got running now, but now what it is is a beam with my pen on it, my little squiggle pen. And you've seen this before. It's a rotating mass. It's a static imbalance. It shakes this beam.

And I've added to it a second little beam. It's a little blade of steal with a heavy magnet on it. That's my m2. This is my k2-- is this little beam. This is the original system, a mass, it's close to a natural frequency, and the rotation rate of the eccentric mass is right at the natural frequency of this beam. So because it's delicate to set up, I've got to show you the system first with the second mass attached to it, and I think I've got it tweaked so that it'll sit there. It's right at the operating point, and you ought to see the second mass moving like crazy, and the original mass, the beam, not moving much at all. So let's see if we can make it work. We're going to need to kill the lights.

And you can see in here the rotating mass going around and around, and so I have the strobe light detuned just a little bit so that you can see the system. It's not quite synchronous. Watch this little white blob out here. You see it going up and down. That's your second mass, and is the main beam-- it's moving a little bit, but not much. It's got a little damping, and so it isn't perfectly down at that 0 point, but very close. So this is the system operating close to that null point.

So now if I remove the dynamic absorber, the second little mass spring system, then the main mass and the beam ought to shake like crazy. Just changing the tuning a little here. So now I'm going to remove the absorber. All right? And now it's going like crazy. So that's an illustration. You can come up with the lights.

Now, you hear beating? This little pen-- this little rotor in here is driven by a little DC motor and a single AA battery, and it's not feedback controlled. Its frequency can vary, and it isn't that powerful. So as this thing starts moving up and down, it actually takes real torque to make that weight go around while this whole system is accelerating up and down. And the motor just isn't up to it, just isn't powerful enough, so the speed changes. This thing can't hold constant frequency. So that's the demo. Questions?

So we've kind of embedded in this lecture-- this lecture was to introduce you to the idea that you can write a transfer function for a multiple degree of freedom system-- has embedded in it all the natural frequencies of the system. When you use a transfer function to calculate the response of the system, you are getting the contributions of all the modes at once. So you essentially solve the equations of motion directly. So you now have seen there's two ways to go about analyzing multiple degree of freedom systems-- the technique known as modal analysis-- one mode at a time and then add it together. Or like with transfer functions, where you're just solving the whole thing at once, and each transfer function has in it all of the information about all of the modes of the system. So if it's a five degree of freedom system, you're going to see five peaks out here. Questions?

AUDIENCE: So this is for [INAUDIBLE]? But initially, since you have a [INAUDIBLE], does it have to start out moving to be an [INAUDIBLE] moving and then it stops.

PROFESSOR: So there's going to be a transient phase, sure. Any system, when you start it up, is going to have transients, and the transients are the equivalent of initial conditions. So displacement and velocity-- and the response of a linear system that vibrates, that has natural frequencies to a set of initial conditions, is vibration at what frequencies?

AUDIENCE: Do you know how they overcome this in helicopters? [INAUDIBLE] that we saw at the beginning of the year?

PROFESSOR: Yeah, the helicopter-- so I want to answer the question. Response to that transience is at the natural frequencies of the system, maybe some combination. The original displacement requires contributions of several modes. You'll get decay contributions at each of the natural frequencies, and once they've died out, this motor has started up. The thing will settle into what we call steady state response, and it will only be at the excitation frequency.

And I haven't show you anything about-- can you solve the equations of motion from start up at time 0 through the whole messy transient phase to steady state? Sure. And we haven't talked at all about that. And actually, for vibration stuff, it is very important. Steady state answer is important. The transient answer is pretty important, and for you to have a feeling for that, you actually know quite a fair amount about the basic concepts of how things vibrate now.

Now, your question about the helicopter-- so if you have a uniform helicopter blade-- they rigged that to do what it did. I don't know what they were experimenting with, but if you do-- there's a problem that I've actually given in a vibration course. If you design a really simple helicopter that has a uniform blade. It's a uniform rod, and it's pivoted at the center, and you spin it up, and if you spin it fast enough-- if you spin it slowly, it will droop because of gravity, but if you spin it fast enough, r omega squared is a lot bigger than g. And the thing flattens out, and the blades are going around and around.

The natural frequency of that system is exactly the rotation rate. A uniform blade pivot at the center, going around fast, has a natural frequency like this. It happens to be exactly at the rate of rotation. So any perturbation will start it doing bad things. Helicopters are never ever designed like that. The helicopter has a rotor disk in here, has a finite radius before the pin, and then the blades are out here. The pin around which-- and that changes the resonance frequency. And of course, you can change the mass distribution in the blade and so forth.

So what they had done in that case to rig it so the thing beat itself to death, I don't know. But there's good design practices with helicopters so that the blade frequency is not at the rotation rate, because at rotation rate, you're going to have a lot of excitation, hitting turbulence in the air and so forth. Yeah?

AUDIENCE: So you mentioned that you use something like this in the Hancock Building, but I would imagine that the wind force on the Hancock Building does not have a uniform frequency all the time. So is the idea that [INAUDIBLE] control based on resonant frequency [INAUDIBLE].

PROFESSOR: I've forgotten the correct name now, even though I've used these many times now. The optimally tuned and damped dynamic absorber is sort of the second level of dynamic absorber design. Typically, a two degree of freedom system looks like that if it's lightly damped, but you can design that second mass spring and a damper so that you can make the transfer function, H11 here, look like this, where you get the two peaks are of equal height, and the worst-- and you can design them so that the worst case amplitude response over here is pretty low. x1 static is square root of 1 over 1 plus mu or something like that.

So the bigger you make this, the better the performance, but you have to have optimum damping, and you have to have optimum tuning. And tuning means, if the there were-- we tune this thing so that k2/m2 was exactly equal to a particular frequency, in this case k1/n1. But for the optimally tuned and damp dynamic absorber, you actually tune things-- it's tuned a little differently. You put damping in it, and you make it behave pretty well over the entire frequency, range rather than almost perfect at one frequency.

And the Hancock Building had its transfer function, so to speak, without the dynamic absorber-- it has many, many peaks, but it had two problematic ones that were very close together, and they were at about 0.8 radians per second, and 0.81, or something. And this was bending, and this was torsion. And at these two resonances-- and the wind would come along, and it would excite both of them. And the one resonance was just the same as this thing. It just bends. And the other resonance was the first mode in torsion.

So if you were standing on one-- and a building cross section is a funny-- the Hancock Building looks kind of like that in cross section, and so it would rotate in torsion and deflect in bending. So it would be rotating and deflecting, and if they were in equal amounts and they're close in frequency, if you're standing up there, what would you feel? They would beat. The motion would beat.

You'd get a lot of motion, because if you are on one end of the building, you'd get a lot of torsional motion out here combined with the bending motion. And then they would cancel, and then the beat would build up again. And so that was what was happening in the building. And they put in two, so on the 58th floor on each end they put in a mass spring system and over here another one so that they would resist the torsion-- actually, really I lined them up the wrong way. They line up this way. So it has two of them.

They can resist both the bending and the torsion. They each weigh 300 tons, and they it's a box filled up with 50 pound lead bricks. And the box is about-- it's been 20 years since I was up there, but what I recall is this like 8 by 10 feet yea tall. And it slides on a pressurized oil film, and the spring on it is a big pneumatic spring, a big air spring, and a computer runs the whole system. It's shut down until the wind gets above 40 miles an hour, and then turns it on. And it's actually kind of self tuning. It can optimize itself, and then it'll sit there, and it's designed to do this, because this building has these two problem frequencies. And so it has been designed to address both of them.

All right, I think I have to get out of here for the next class. See you next Tuesday. Last lecture.