PROFESSOR: Review model analysis. I figured most-- I figured lots of students have had questions and we could use a little practice on this. So that's what we're going to do.

AUDIENCE: Solving the q thing?

PROFESSOR: Huh?

AUDIENCE: Solving the q function?

PROFESSOR: Yeah, solving the whole thing. So but let's start. I'd still start with the usual concepts. I haven't even erased what we had last time. But this is what the students in the last group said about key concepts for the week. If you want to add something to it, speak up. But modal analysis, multi-degree of freedom, transfer functions, and then model analysis-- more specifically, response to initial conditions, response to steady state harmonic inputs. Anything else that was significant, conceptual, and new in the last week?

Add to that. And then the second thing, what are issues that are muddy for you? Not quite clear, things you want to learn more about, have questions about. Last class. was more on transfer functions for multi-degree of freedom systems and solving initial condition problems is what one student was interested in about from using modal analysis. But do any of you have questions about things that you want to practiced on?

AUDIENCE: So I understand how to get the new mass and spring whatever matricies for the q system, so I can set it up as the mq bq kq, but from there, I don't know how to go back and solve for x, or I don't know how to solve for x.

PROFESSOR: Right. So we'll go through that. We're going to run through a complete modal analysis today, all the steps that you need to do to make it happen. So anything else that you've got a question about that I might be able to get to?

OK, let's get rolling. This will take a little while. So problem for the day is, if you recall last time, we had this demo of this double pendulum. But now we're going to take that double pendulum and make the masses unequal. So well, the masses are equal here, but they could be unequal-- M1, M2, they're each half a kilogram. But we're changing the lengths a little bit, 1.1 for L1 and 1.0 meters for L2. So slightly different lengths, and that'll make this system not symmetric so it won't have 1, 1 and 1 minus 1 mode shapes.

A little weak spring in the middle, possibility of having some dashpot here connected to a non-moving wall, another dashpot here. And the possibility of having harmonic excitations-- F1 on this one, F2 on that one. The whole system's been linearized. The equations in motion look something like this, mass damping matrix, stiffness matrix. Of course, it has gravity terms in it as well as the spring terms. And it's been linearized.

This equation in motion, is it a force equation or a moment equation? Are they force or moment or both? We can have mixed ones like the cart. Problem with the pendulum has one force equation and one equation with units of torque, this one has units of what?

AUDIENCE: Torque.

PROFESSOR: Yeah. This is moments about point A and moments about point B and give you the two equations. So this is a torque equation. So you need to look and see if things inside here make sense. So the MGL, the non-linear equation, the restoring torque, is MGL sine theta. Linearized, it's just theta. So MGL 1 theta is the torque on the first mass. M2 GL 2 theta is the torque on the second mass, and so forth.

And K1 L1 squared, why the L1 squared? This term here gets multiplied by theta 1, so what is that? What's K1 L1 squared theta 1? What kind of a torque-- what is that? Does it make sense? Are its units correct?

So first of all, what's K1 L1 times theta?

AUDIENCE: Force.

PROFESSOR: Yeah, it's a force. What's L1 times theta?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Physically, that is what?

AUDIENCE: Distance.

PROFESSOR: That's the distance that that thing moves, right? And a displacement times a spring constant gives you a force, and then that force times a moment arm gives you a torque. So it makes sense. OK, so L1 squared theta 1 is the torque about point A caused by a displacement theta 1, assuming this one is 0 when you do it. That's the torque caused by a displacement of theta 1 only.

OK, all right. So that's our equations of motion linearized. This is the mass matrix. This is the stiffness matrix that you would get if you go into there and substitute in these values. Yeah?

AUDIENCE: The matrix on the right would be to one of these [INAUDIBLE] on the bottom?

PROFESSOR: This one here? This is a two by two matrix.

AUDIENCE: Oh, OK.

PROFESSOR: And it has to be. It's a two degree freedom system. This is K11, the K11 term. This is the K12 term, the K21 term, and the K2 term.

AUDIENCE: So there's a space between--

PROFESSOR: Yeah, it got a little squeezed here. This space, there's an M2GL2-- and that shouldn't be squared-- plus KL2 squared.

AUDIENCE: Plus [INAUDIBLE]?

PROFESSOR: All right, pretty sure that's what it out to read like. MGL2 KL2 squared. And then each of these are minus KL1L2 minus KL1L2. All of this, this is all in a handout which will be put up on the Stellar website. And just so in terms of reviewing for final and things to go look at, for almost every recitation we've done this during Professor Gossard, the other recitation instructor, has written up the complete problem that was discussed with its solution and put it on the website. So you don't even need to have to copy this stuff down. It's all posted.

In each recitation, we've essentially done the problem that is sort of the objective lesson for the week. So they're a good place to go review. So this whole problem will be up on there. And we found a mistake, actually. And I think it will get fixed before it gets put up, but I think this had a zero in here, and that's actually wrong, or if something got transposed in writing. That's the correct mass.

OK, so you have these numbers. I don't want you to work this stuff out, because I want to focus on the model analysis. So let's see here. I'm going to ask you a question first. So we're going to begin this. Yeah?

AUDIENCE: Is there a torsional damper at the top of the page?

PROFESSOR: Oh, somebody asked a question in the last class and I drew that up there. But if you put a torsional damper up there, some CT value, how would it appear in the equation of motion? Would it have L1, L2 squareds in it? No, it would just simply disappear as a plus CT theta 1 or a theta 2, wherever it's applied directly, theta 1 dot.

OK, so question. Write down your piece of paper-- so I'm not going to make you go to the board today, but I want you to take a minute and write things down. And then we'll check and see if everybody agrees. The reason we can do modal analysis is because of something we called the modal expansion theorem. It's basically the fundamental statement that says we can do this.

So what is the modal expansion theorem? You can write it down mathematically if you want this, just as a little linear algebraic expression, or you could write it out in words. So take 30 seconds and write down what makes modal analysis work, what basic proposition.

All right, somebody help me out. What is the modal expansion theorem? What's it say?

AUDIENCE: Any motion of the system can be described as a weighted sum of the natural modes?

PROFESSOR: Weighted sum of the motions of each of the natural modes, OK. So that's the statement, the most succinct way to say it is in the original generalized coordinates, you can express them as uq. u is the matrix of what?

AUDIENCE: [INAUDIBLE]

PROFESSOR: No, this is--

AUDIENCE: Oh no, of the mode shapes.

PROFESSOR: Mode shapes. And the q's are the individual modal coordinates, right? And so this, expanded, says that this two degree freedom system has two generalized coordinates-- theta 1 and theta 2. And the response of either of the actual motion of the system expressed in the generalized coordinates can be made up as the sum of each of the modal coordinates, the q sub i's, scaled to this shape the mode shape for that mode.

So this multiplies the mode shape, and that-- so everything contributed by mode one will move in the shape of mode one, and that will be reflected in the motion of the generalized coordinates. And in this case, it's a two by two. It's a two-degree freedom system. Here's the mode shape of mode 1 times q1, which we're going to solve for, plus the mode shape of mode two times its model motion. OK, that's the model expansion theorem.

This allows you to-- in order to do this, you have to solve for these qi's. So what is the equation of motion that governs the behavior of each of the modal coordinates? Write it down. What's the whole reason we do this? There's one particular equation that every one of them satisfies.

So equation of motion, not asking for solution. I just want equation of motion that governs these modal motions. I think I'll give you a minute to think about it.

OK, somebody help me out. What's the equation of motion that I can write that will describe the motion of any one of these modal coordinates, qi? Christina?

AUDIENCE: With the fancy M's and C's and K?

PROFESSOR: Pardon?

AUDIENCE: With the fancy M's and C's and K?

PROFESSOR: Well yeah, but it's basically a very simple equation of motion, which you should be familiar with by now. What's it look like? For any one of these modal coordinates, what is the equation of motion that governs it? Why do we go to all this trouble? There's a reason for doing this. It's because--

AUDIENCE: Single degree of freedom.

PROFESSOR: Ahh, the single degree of freedom oscillator equation, right? This is the reason we do this. Single degree of freedom systems are mathematically simple. You've seen them since high school. You've seen them in 1803. It's the second order linear differential equation that looks like this. And you already know everything there is to be known about that equation. And that's why one of the reasons why we do this.

You don't have to solve a complex set of simultaneous differential equations, you just have to know one. And so the ith one, you need to know the ith modal mass, the ith modal damping, the ith modal stiffness, and the modal force. And we've-- in this course, we've taught you how to solve two kinds of single degree of freedom system problems. One is response to initial conditions when the forces on the right hand side is zero, and the other is this steady state response to a harmonic input, so a cosine omega t kind of input. So that's what we focused on in this course, because it's vibration we're interested in.

So we've solve this equation for two kinds of problems. Now to do modal analysis, you need to be able to find these quantities. These we called the modal masses, the modal damping coefficients, the modal stiffnesses. How do you get those? How, for example? Write down on your paper, what equation, what linear algebra thing do you have to work out to get the modal masses for this system?

Let's say all of them. I want a two-degree system. What's the state? What's the mass, the linear algebra you have to work out to get the modal masses? Somebody help me. What is it? Yeah?

AUDIENCE: Transpose of the mode vector-- sorry, the mode matrix multiplied by the mass vector?

PROFESSOR: Yeah?

AUDIENCE: And then multiplied by the mode matrix.

PROFESSOR: All right. All right. The modal forces are u transpose f. The modal masses-- and I've drawn these little diagonal marks in here to remind, you these matrices become all diagonal when you do the model analysis. Coordinate transformation, I'll call it. So the modal masses are u transpose mu. And this m is a matrix, and it's the original mass matrix of the system.

Modal stiffness, matrix u transpose ku. And the modal damping matrix, u transpose cu. But this one can be problematic. You have to force this one to behave. These are guaranteed to behave, all right? So for this problem, and I know some of you are a little rusty calculating these things. So there's the model mass matrix. And here is the modal matrix of eigenvectors or mode shapes of the system. u is made up of columns, and each column is one of the mode shapes.

The convention is to order them from the first mode to the nth mode where the order is established by the natural frequencies. The lowest natural frequency is first-- second, second, up to the highest natural frequency. Anyway, here's mode one, here's mode two for this system.

You have to choose a way in which to normalize the mode shapes. I choose to normalize them usually. I say, I'm going to make the top element of them one. And I do that whatever-- if do MATLAB like you do, there's a function called Eig, which means eigenvalue. You do Eig of A, it'll give you the eigenvalues of matrix A. And it'll give them back to you unnormalized and unordered.

Well, so you can write a little program to put it all in nice order. But if MATLAB gave you back the mode shapes for a system and it said, well, the mode shapes of the system are-- and it's a two by two system. 2 and 0.4 and 0.6 0.5, you know that the vectors are the mode shapes, the columns. How would you normalize those? How would you make the top element 1 in this first one?

AUDIENCE: Divide by 2.

PROFESSOR: Divide what by 2?

AUDIENCE: The Entire column.

PROFESSOR: The entire column by 2. Just factor 2 out. So this would become 2/2 and 0.4/2. And then that's 1 and 0.2. So you just normalize this vector so the top element is 1. So do you have to normalize it? Could you use them this way? Sure. But once chosen, once the normalization is chosen, the key to doing modal analysis is you have to stick with it. You can't move. You can't mess with that halfway through, or you totally screw up the solution.

So you pick your normalization. When you calculate the natural frequencies and mode shapes, you pick a normalization, and you must ride with that all the way through, including putting it back together here at the end, this summation. OK, so let's do-- I want you to do this computation. Calculate the modal masses for this problem. That means you have to remember what a transpose is. There's the model, that's the model matrix. And the modal mass matrix is right there. So actually, just do the arithmetic. Take a few minutes. Yeah?

AUDIENCE: On the exam, we won't have calculators, so like is it all going to be variables, or--?

PROFESSOR: Say that again?

AUDIENCE: On the exam, we won't have calculators or anything.

PROFESSOR: On the exam, we'd either make it so simple that you can, in fact, do it in your head or on paper, or we won't ask a question that you have to do it that way. Or we'll accept an answer where you put it down but don't have to multiply it out. So just do this one, just to see if you remember the mechanics of doing the linear algebra to get that.

OK, somebody have an answer for me here for the modal mass matrix? What's the first element? Somebody help me. Give me a number and then everybody else can check you.

AUDIENCE: 0.254.

PROFESSOR: Say again?

AUDIENCE: 0.254.

PROFESSOR: 0.254. OK, what about the second one, this element over here? Speak up.

AUDIENCE: 0.

PROFESSOR: Yeah, it better be 0. What about this one down here? All right, how about this one?

AUDIENCE: Wait, that first one's not right.

PROFESSOR: Hmm?

AUDIENCE: That first one's not right.

PROFESSOR: OK.

AUDIENCE: [INAUDIBLE]

PROFESSOR: So did you give me the first one? So you're authorized to change this. OK, you got point what?

AUDIENCE: 859.

PROFESSOR: 8598, And this 0, 0-- how about this second one down here now?

AUDIENCE: 3.48.

PROFESSOR: 3.48. Anybody else get anything different? So let's-- we have 605 00 and 0.5, and we're multiplying that.

So if you're looking for just one of them, by the way-- where'd my eraser go?-- all you need is one of the modal masses. The only ones that give you non-zero results is when you compute u transpose for mode r m u for mode r. So if you're only looking for this second one, you only have to do the calculation for that mode.

So this then becomes a set of matrix, matrix, matrix, you only have to do a couple of vectors. So this looks like for mode 2, it's minus what? Mode 2 is 1, and minus 1.6949 times 0.60500 0.5 times 1 and minus 1.6949. So to get just one modal mass, this is M2. To get just one modal mass, now you only have to do that calculation. You only have to do the computation using one of the mode shapes.

So it's this times that, and this times that gives you some numbers back. 0.605 and half of this about point 0.8, and then you take that and multiply again. Anyway, can somebody give me this second number? I have a 3.48. Anybody get anything different? Pardon?

AUDIENCE: I messed it up.

PROFESSOR: OK. There are about 18 of you and nobody can do this calculation?

AUDIENCE: 2.13.

PROFESSOR: Say again?

AUDIENCE: 2.130.

PROFESSOR: All right, I have a 2.132 and a 2.0. OK, there you are. OK, so if you were having trouble sorting that out, probably a good thing to go back and review a little bit of your linear algebra. OK, if you do u transpose ku, you get this. Yeah?

AUDIENCE: What is it for the [INAUDIBLE] you have a multiplied by F1 L1, not just F1?

PROFESSOR: OK, I'm going to guess where I'm going next. So to get the stiffness matrix, u transpose ku, you get this. We're going to leave the damping matrix for a minute. We need that. We need the modal excitations.

So I'm just going to do a particular problem. I'm going to say, let's let F2 be 0. We'll only have one force. And the first force will be F1 cosine omega t. And so now I need to do u transpose F. So here is u transpose. Here's the F's, and I guess I've got to keep my cosine omega t here. So you multiply that out. What do you get? Yeah?

AUDIENCE: Why is F1 L1 [INAUDIBLE]?

PROFESSOR: Well, because F1 is just the applied force, but the equation in motion that we're working with, if you go back and look at it, what is the forces on the right hand side? The forces have to be moments, right? If we're putting a force down there, it's a moment equation. We need the moments about the pivot. So it's the force times L1 or the force times L2.

AUDIENCE: Oh, so we don't have to use actual magnitude of 4. We have to use [INAUDIBLE].

PROFESSOR: You have to. You well eventually-- well, these kind of problems are easiest to do once you reduce them to numbers. I'm leaving the force in it as a variable at the moment just so you can see how it carries through the problem. But I'm just saying, in the real problem there, let's say there is no F2. There is an F1, and the F1 of t looks like a magnitude f1 times cosine omega t. That's the only force I have in the system.

But we're working with equations of motions. An equation of motion is the right hand side-- F1, L1, and F2 L2. And you have to retain the L1's and L2's in order to have the correct equation of motion. So when we say this is kind of just a generic form, this is the modal force vector is the mode shape matrix times the modal excitations in the original coordinates. So I just wrote F here, but what this really means is this is F1, L1. F2, L2.

These are the real generalized forces in the system. OK, they're the real, generalized forces. Now, I've let F2 be 0. So the only generalized force is F1, L1 cosine omega T. I multiply that by u transpose to get-- and what do I get? This is a pretty simple calculation. So this becomes F1, L1 cos and F1, L1 cos. So the two modal forces are identical. Yeah?

AUDIENCE: Why do we know this one so we don't have to include a phi?

PROFESSOR: Ah, well the phi doesn't come out until the answer. What does the fee mean? What's that phase angle mean? If you-- remember, we're doing steady state problems in which F1, L1 for example, if it's cosine, we're just assuming it looks like that. And we're looking for a solution of theta 1.

And actually, we can't quite go there yet. We're doing single degree of freedom problems, right? We are looking for a solution for Q1. We turn this into a modal force, q, but it happens to be the capital Q1, the modal force, is F1 L1, right? Cosine omega t. So that's the input. The output is q, little q, of t, the modal coordinate. And what does it look like? We're only doing steady state, no transience.

It looks like a response that looks like this, but it's shifted. I'll draw this so its peak is right here, cosine is its-- it's shifted in time by this amount, between this peak is here versus the peak being there. And that we can represent as a phase angle. Remember, one period from here to here is 2 pi radians. So some portion of that period is an angle. You can interpret it as an angle or you can interpret it as a time delay.

And this then has the form of some q1 magnitude cosine omega t minus that phase shift. So it's only-- that's the only thing the phase shift means. Cosine in doesn't mean exactly the response out's going to be exactly in the same perfectly in time with it. It could be shifted.

Now you know that it-- for a single degree of freedom system at resonance, what's the phase angle? Do you remember that? It's always one number. It's pi over 2. A shift of pi over 2, if you remember your trigonometry, takes you from cosine to sine or sine to cosine. It's trying to tell you that the response is shifted by exactly a quarter of a cycle, pi over 2. And the reason for that is that at resonance, all of the excitation is going into overpowering the damper.

And the damper's motion is proportional to velocity. And if velocity, if displacement looks like cosine, velocity looks like one derivative of it, which is sine. So that pi over 2 says that the response velocity is in phase with the force, and that makes sense. Something I hadn't said in lecture but I really meant to is I want you to think about something here. Single degree of freedom system, we'll even write the one we're working with.

M1 Q1 double dot plus C1 Q1 dot plus K1 Q equals some F1 L1 cosine omega t, and I'm going to let omega E be at omega 1. But this is the equation of motion, right? Let's plug in. We're saying we're going to do this right at the-- and we know the response of this is Q1 is some magnitude cosine omega t minus the phasing. We know we can plug that in. We get minus M1 omega squared Q1 plus K1 Q1 minus C1 omega Q1. And this one goes like sine omega T minus phi.

This one here needs to get multiplied. This term gets multiplied by cosine omega t minus phi. You plug that into this. This term and this term both behave like cosine. This term, one derivative behaves like minus sine. One derivative of cosine gives you minus sine, right? And when omega equals omega 1, so when you're right at resonance here, what happens?

This is squared. And put the squared down here, there we go. So this is omega. But now I'm going to let it be right at omega 1. What is omega 1 in terms of K's and M's? K1/M1, otherwise, one of the checks you can make when you finish doing your modal-- if you take that modal mass and the modal stiffness and you divide 7.96 by 0.8598, that had better be omega 1 squared. That's a good way to check that you've done all your arithmetic right.

All right, I'm going to plug in omega 1 squared here equals K1/M1. So I put in, this becomes minus M1 K1 over M1, which is minus K1, right? Hmm, plus K1. That resonance, this term accounts for the inertial force in the system, the force required to accelerate the mass. This is a force equation. This accounts for the force required to push the spring.

The amazing thing that happens is at resonance, the inertial forces exactly cancel the spring forces. And the equation of motion reduces to minus C1 omega 1 Q1 sine omega 1 t minus a phase angle equals, in this case, F1 L1 cosine omega 1 t. So how to satisfy that equation? What phase angle will satisfy that equation? Has to be pi over 2.

And if you put pi over 2 in here, this minus sign turns into plus cosine. And you're left with C1 omega 1 Q1 equals F1 L1. So all of the exciting force goes into pushing the dashpot. So that's why you get the big peak in the transfer function. It takes in no force to move the spring. It takes no force to accelerate the mass. They exactly cancel. And all the force is available to drive just the dashpot, all right?

So let's move on now. We need to get to our answer here. So the last piece of this is we now know the modal forces, and your assignment is to let's let omega-- let's see, where's my-- did I do this somewhere? I guess not. I guess I erased it, so we can pick anything we want.

Let's let omega equal omega 2. We're going to drive this thing at the natural frequency of the second mode. That's the excitation. What do you expect? Which mode do you expect to dominate the response?

AUDIENCE: The second.

PROFESSOR: The second mode. Why?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Because you are driving it. You know it's got a transfer function that looks like this, and you're driving it right here. And where are you driving it? If you're doing that, where are you on the transfer function for the first mode? Omega 2, it's a little higher than omega 1, but not a lot. So if this were the first mode's transfer function, where would you be driving it on this transfer function? If this is omega over omega 2, you're over here a little bit. You're driving it a little bit higher than the natural frequency of mode 1.

This would be omega 2 over omega 1 on that. So one's sitting here, one's sitting there. Which one's going to dominate? The big one, OK? Because they have equal modal forces. They both happen to be F1 L1. OK, so how do you do that? So the last step in this thing is I want you to, for this case, find the first-- no, let's do this. Find the second mode contribution to the response in the original generalized coordinates.

A quiz would often be written this way. It's trying to make it easier for you. I'm asking for only one mode's contribution. What does that actually mean in the original statement of the modal expansion theorem? We know the total response looks like this, right? I'm asking you to give me only the second mode's contribution. What am I asking for? Just this term. So I'm telling you, you don't have to bother with the other term to satisfy me. Just tell me what this one is, because I know this is one is going to be the dominant one.

OK, so how do I do that? So for this problem, what is the steady state response of this due to mode 2 only? So mathematically, just in-- what's that look like over here? Just that second term, right? It's the modal, mode shape vector for mode 2 times Q2 of t. So you need to tell me how to find Q1 of t. It's a single degree of freedom problem excited by steady state excitation.

AUDIENCE: It's like a transfer function.

PROFESSOR: Ah, magic word-- transfer function. So we need the magnitude of the-- this is a linear problem, so the response is linearly proportional to the--?

AUDIENCE: Force.

PROFESSOR: Force. So the magnitude of Q1 times a transfer function that looks like the hx over f transfer function. In this case, we call it the magnitude of HQ2 per unit Q2. And that multiplied by-- it looks like cosine in this case-- omega 2 t minus some phi 2. That's what we're looking for. What's this? Tell me what that looks like. Yeah?

AUDIENCE: Shouldn't it be Q2?

PROFESSOR: Thank you. So in effect, what's the transfer function? I want it in all its detail now. That's the magnitude of the force. What's in the numerator of this? Numerator.

AUDIENCE: Well, it has parentheses around it.

PROFESSOR: Yeah, it's in the denominator though. I want just the numerator part. This transfer function expression for a single degree of freedom system, what's in the numerator?

AUDIENCE: 1/K.

PROFESSOR: 1 over which K? K2. Modal K2, We're now in the modal system. And in the denominator of that transfer function, what's it look like?

AUDIENCE: 1 plus omega.

PROFESSOR: 1 minus omega squared over-- in this case, omega 2 squared, squared, plus--

AUDIENCE: 2 [INAUDIBLE].

PROFESSOR: Zeta.

AUDIENCE: Zeta, yeah.

PROFESSOR: 2.

AUDIENCE: 2 omega over omega 2 quantity squared.

PROFESSOR: There we go. Now in this problem, what is omega? So that makes this a 2, this a 2. This term, what happens to it? 1 minus 1. This term, this goes to 1. This is 2 zeta quantity squared square root. Just 2 zeta. So this whole at resonance, any one of these single degree of freedom systems that are at resonance, the response is the magnitude of the force-- in this case, it's positive F1 L1-- over K2 times 1/2 zeta 2.

So F1 L1 is the modal magnitude of the moral force divided by K2 gives you what we call the static displacement of the system. And this is the dynamic application. So oftentimes on quizzes, you're asked to do the response at resonance, because it makes all this algebra so simple. It boils down to 1/2 zeta. So the only thing left to do is we need the damping ratio for this system.

So rather than do that, so now that, we're missing something yet. We're missing-- that whole thing gets multiplied by cosine omega 2 t minus the face angel. What's the face angel? Pi over 2. So Q2 of t is F1 L1 over K1 1/2-- whoops, F1 L1 over K2, 1/2 zeta 2 cosine omega 2t minus. And at the very-- how do we get back to generalized coordinates, theta 1 and theta 2?

AUDIENCE: Stay here.

PROFESSOR: Right here. And we've computed that. This part, you multiply it by the mode shape. The mode shape partitions out the response in the right amount to coordinate one and the correct amount to coordinate two.

AUDIENCE: For the damping ratio, would you take the modal, like, mass and [INAUDIBLE]?

PROFESSOR: In reality, what you do with damping ratios is you're working with real things out there in the real world. If you can, you go up and give the thing a kick and get your stopwatch out and say, how many cycles does it take to the k? And is it light damping or not? If it vibrates a lot, it's usually light damping. And if it's light damping, you can force this damping matrix. You can just make-- force it to behave, even if it isn't perfect.

And it is a perfectly adequate, useful answer. Even if it isn't perfectly diagonal, it just doesn't matter when it's light damping. And so in this problem, what you do if you go estimate the damping for the system. You say eh, it looks to me to be about 2% for mode one and 1.5% for mode 2. And you just say, how can I fit? How can I represent the damping in the system?

And one of the easiest ones is to say that the original damping matrix is some alpha times the mass matrix plus beta times the stiffness matrix. And in this problem-- or you can use any part of that. And if you're only trying to match one mode, see, this problem it's-- this system is being driven at the natural frequency of one mode. That mode is dominating the response, right? So we really actually only need a good model of the damping for that mode.

Even if you have the completely wrong damping for the other mode, it will have little effect on its answer because you're not at resonance. When in the transfer functions, which look like this, at resonance, this happens. Then you find out that the only force resisting the input is the dashpot. At low frequencies, you find out that over here, the dominant force is what it takes to move the spring. And the damping isn't very important.

And so even if you're wrong by 50%, it just doesn't-- 50% of a little bit compared to what it takes to move the spring is not a big deal. And over here, it behaves like the mass. Out here it's called the mass controlled region. Over here is the stiffness controlled region. And in the vicinity of the peak is the damping controlled region. So at low frequency, this term dominates. At high frequency, that term dominates. And at resonance, this is the dominant term.

So in this case, let's let the damping be some alpha times the mass matrix. Then when you do UTCU, you get alpha times the modal mass matrix. And therefore, C2, which is the one we care about, is equal to alpha times M2. And we have M2, our modal mass, 2.04, right? So that says this is equal-- C2 is equal to alpha times 2.04.

And if I want-- that's C2. And how do I get zeta 2? Zeta 2 is C2 over 2 omega 2 M2. That's just the definition of the damping ratio. I know this. I know this. So this is going to be alpha times 2.04 over 2 omega 2 M2. And I've measured it. I've taken it, and I know this is 0.02. About 2% damping. Solve for alpha.

You now have the whole thing that you need. You can now find it. That's all you need. Yeah?

AUDIENCE: Where do you get the alphas and omegas from again? Alphas and omegas.

PROFESSOR: This is simply-- this is called Rayleigh damping. Lord Rayleigh 150 years ago came up with this. And he just said hey, by the way, if you model damping this way, you can automatically make the equations of motion, u transpose CU, go diagonal. And you have a two parameter model with which you can juggle them to make any two damping ratios of the system be exactly what you want them to be.

So it's just-- if I left this as alpha N beta, then I would have worked this problem as-- this would have been an alpha M plus a beta K. C2 would have been an alpha M2. But now it would be equal to alpha M2 plus beta K2. And then this still applies, except that it'd have this alpha and a beta. And you could do it for the other equation. You could do it for zeta 1. And you'd have two equations and two unknowns. You solve for alpha and beta.

But think about what I just did here. If I made a measurement of the system, I said the damping for mode 2 is 2%. Do I have to go through all this junk? No, because I know the answer looks like that. All I have to know is what the damping is. And that's my approximate solution for the problem. This is just how to satisfy all the mathematics if you want that perfect mathematical model for which you can write out u transpose CU and get them. Well, this is one way that you force the damper matrix to have the properties that you want it to.

But in reality, you just measure the damping and put it in the answer. Very good, this is our last go around at recitations. See you in class on Tuesday. We'll do something fun that's not covered on the final exam. I'll give a little review of what's going to be on the final, a list of what's on it. And we'll talk about strings and beams and things that apply to pianos and violins and so forth.