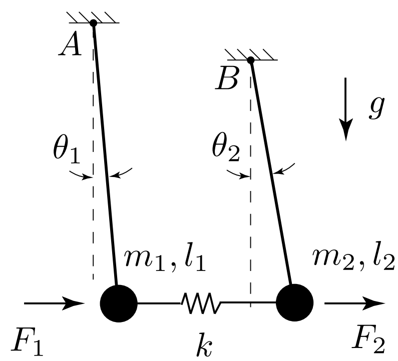
2.003SC
Recitation 12 Notes: Modal Analysis

Modal Analysis of Double Pendulum System - Problem Statement

Consider a double pendulum consisting of two masses and one spring as shown in the figure below. Note that θ_1 and θ_2 are small-angle displacements.



The system's equations of motion are,

$$\begin{bmatrix} m_1 {l_1}^2 & 0 \\ 0 & m_2 {l_2}^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_1 {l_1}^2 & 0 \\ 0 & c_2 {l_2}^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_1 g l_1 + k {l_1}^2 & -k l_1 l_2 \\ -k l_1 l_2 & m_2 g l_2 + k {l_2}^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_1 l_1 \\ F_2 l_2 \end{bmatrix}$$

or

$$\underline{M}\ddot{x} + \underline{C}\dot{x} + \underline{K}x = \underline{F}$$

For $m_1 = m_2 = 0.5$ kg, $l_1 = 1.1$m, $l_2 = 1.0$ m, $k = 0.5$ N/m, the M and K matrices above are

$$\underline{M} = \begin{bmatrix} 0.605 & 0 \\ 0 & 0.5000 \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} 6.005 & -0.5500 \\ -0.5500 & 5.4050 \end{bmatrix}$$

and the natural frequencies and natural modes are,

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 3.0445 \\ 3.3851 \end{bmatrix} rad/sec \qquad \underline{U} = \begin{bmatrix} 1 & 1 \\ 0.7139 & -1.6949 \end{bmatrix}$$

Assume that the system damping matrix is of the form

$$\underline{C} = \alpha \underline{M}$$

Assume that the external forces are

$$\begin{bmatrix} F_1 l_1 \\ F_2 l_2 \end{bmatrix} = \begin{bmatrix} 0.2cos(\omega_1 t) \\ 0 \end{bmatrix} \qquad [N - m]$$

- Find $\underline{U}^T\underline{MU}$, $\underline{U}^T\underline{KU}$ and $\underline{U}^T\underline{F}$ (the modal forces in terms of $F_1 l_1$.)
- Check the natural frequencies
- Find α such that the damping ratio for mode 2 is 0.04 .
- Find the steady-state response in terms of the generalized coordinates θ_1 and θ_2, due to the contribution of the first mode only.

Modal Analysis of Double Pendulum System - Solution

We seek uncoupled equations,

$$\underline{U}^T\underline{MU}\ddot{\underline{q}} + \underline{U}^T\underline{CU}\dot{\underline{q}} + \underline{U}^T\underline{KU}\underline{q} = \underline{U}^T\underline{F} = \underline{Q}$$

$\underline{U}^T\underline{MU}$, $\underline{U}^T\underline{KU}$ and $\underline{U}^T\underline{F}$

$$\underline{U}^T\underline{MU} = \begin{bmatrix} 0.8598 & 0 \\ 0 & 2.0414 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

$$\underline{U}^T\underline{KU} = \begin{bmatrix} 7.9699 & 0 \\ 0 & 23.3918 \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

$$\underline{U}^T\underline{F} = \begin{bmatrix} 1 & 0.7139 \\ 1 & -1.6949 \end{bmatrix} \begin{bmatrix} F_1 l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 l_1 \\ F_1 l_1 \end{bmatrix}$$

Damping

When the system damping matrix is of the form

$$\underline{C} = \alpha\underline{M}$$

$\underline{U}^T\underline{CU}$ will be a diagonal matrix.

$$\underline{U}^T\underline{CU} = \begin{bmatrix} \alpha M_1 & 0 \\ 0 & \alpha M_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

We have two independent SDOF equations of motion

$$M_1\ddot{q}_1 + C_1\dot{q}_1 + K_1 q_1 = Q_1 = F_1 l_1$$

$$M_2\ddot{q}_2 + C_2\dot{q}_2 + K_2 q_2 = Q_2 = F_1 l_1$$

So

$$\zeta_1 = \frac{C_1}{2\omega_1 M_1} = \frac{\alpha}{2\omega_1}$$

$$\zeta_2 = \frac{C_2}{2\omega_2 M_2} = \frac{\alpha}{2\omega_2} = 0.04$$

So

$$\alpha = 0.04(2\omega_2) = 0.271$$

$$\zeta_1 = \frac{\alpha}{2\omega_1} = \frac{0.271}{2(3.0445)} = 0.0445$$

Last bit

So,

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \underline{U}\underline{q} = [\underline{u}]^{(1)} q_1(t) + [\underline{u}]^{(2)} q_2(t)$$

$$q_1(t) = |q_1| cos(\omega_1 t - \phi_1)$$

$$|q_1| = |Q_1||H_{q_1/Q_1}(\omega)|$$

$$|q_1| = |Q_1| \frac{\frac{1}{K_1}}{[(1-\frac{\omega^2}{\omega_1^2})^2 + (2\zeta_1 \frac{\omega}{\omega_1})^2]^{\frac{1}{2}}}$$

$$|q_1| = \frac{Q_1}{K_1}\frac{1}{2\zeta_1} = \frac{F_1 l_1}{K_1}\frac{1}{2\zeta_1}$$

$$|q_1| = \frac{0.22}{7.9699} \cdot \frac{1}{2} \cdot \frac{1}{0.04} = 0.345$$

$$\phi_1 = \frac{\pi}{2}$$

$$q_1(t) = 0.345 cos(\omega_1 t - \frac{\pi}{2})$$

The contribution of the first mode only is

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.7139 \end{bmatrix} 0.345 cos(\omega_1 t - \frac{\pi}{2})$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0.345 \\ 0.2463 \end{bmatrix} cos(\omega_1 t - \frac{\pi}{2})$$

The contribution of the second mode is

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.6949 \end{bmatrix} q_2(t)$$