Solution to Problem 1, Problem set 11

a. The equations of motion may be constructed by considering the free body diagrams of each rotor in torsion about the long axis of the rotors, the z axis. For now, no damping is considered. Summing external torques about the z axis on the first rotor leads to the following equation:

$\sum \tau_z = I_{zz,1}\ddot{\vartheta}_1 = -k_{t1}\vartheta_1 - k_{t2}(\vartheta_1 - \vartheta_2)$. Summing external torques about the z axis on the second rotor leads to:

$\sum \tau_z = I_{zz,2}\ddot{\vartheta}_2 = -k_{t2}(\vartheta_2 - \vartheta_1)$.

When put in matrix form these equations of motion become:

$$\begin{bmatrix} I_{zz1} & \\ & I_{zz2} \end{bmatrix} \begin{Bmatrix} \ddot{\vartheta}_1 \\ \ddot{\vartheta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (1)$$

Each mode will oscillate at a specific natural frequency and with a particular mode shape such that we may assume that the undamped, free vibration solution of these equations of motion will be of the form:

$\begin{Bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos(\omega t)$. By substituting this into the matrix for of the equations of motion, we obtain the following:

$$\left[-\omega^2 \begin{bmatrix} I_{zz1} & \\ & I_{zz2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \right] \cos(\omega t) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (2)$$

The time dependent cosine term may be cancelled out leaving the algebraic equation that follows:

$\left[-\omega^2 \begin{bmatrix} I_{zz1} & \\ & I_{zz2} \end{bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \right] \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. Adding the two inner matrices yields:

$$\begin{bmatrix} -\omega^2 I_{z1} + k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & -\omega^2 I_{zz2} + k_{t2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (3)$$

The rules of linear algebra tell us that to satisfy such an equation requires that the determinant of the matrix on the left must be zero, which results in:

$\left(-\omega^2 I_{zz1} + k_{t1} + k_{t2}\right)\left(-\omega^2 I_{zz2} + k_{t2}\right) - k_{t2}^2 = 0$. This leads to a 4th order equation in ω.

$$\omega^4 I_{zz1} I_{zz2} - \omega^2\left(I_{zz2}k_{t1} + I_{zz2}k_{t2} + I_{zz1}k_{t2}\right) - k_{t2}^2 = 0. \quad (4)$$

Defining $s = \omega^2$, simplifies this to a quadratic equation in the variable s, which may be solved for the two roots for s, using the quadratic equation. This leads to two solutions for ω^2, the natural frequencies of the system. This is completely general for any values of inertia and stiffness for this system.

It is easiest to substitute in values for the physical parameters before using the quadratic equation to find the natural frequencies. Let I_{zz1}=0.2 kg-m^2 and k_{t1}=4000 N-m/radian. Then because $I_{zz1}/I_{zz2}=k_{t1}/k_{t2}=10$, I_{zz2}=.02 kg-m^2 and k_{t2}=400 N-m/radian. Substituting these values in the characteristic equation above leads to:

$$0.004s^2 - 168s - 1{,}600{,}000 = 0 \qquad (5)$$

Use of the quadratic equation provides solutions for the roots

$$\Rightarrow s_1 = \omega_1 = 120.82 \text{ radians/sec}$$

$$s_2 = \omega_2 = 165.54 \text{ radians/sec}$$

Hand solution for the mode shapes is done by a substitution of the now known natural frequencies back into the algebraic equations shown in Equation (3) and solving for a_1 and a_2. This is done one natural frequency at a time.

$$\begin{bmatrix} -\omega_1^2 I_{z1} + k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & -\omega_1^2 I_{zz2} + k_{t2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$ yielding two equations. The first is

$\left(\dfrac{-\omega_1^2 I_{z1} + k_{t1} + k_{t2}}{}\right) a_1 - k_{t2} a_2 = 0,$ which can be solved for the ratio, $\dfrac{a_2}{a_1} = \dfrac{-\omega_1^2 I_{z1} + k_{t1} + k_{t2}}{k_{t2}}$

Substitution of real values for the natural frequency and system properities leads to:

$$\frac{a_2}{a_1} = 3.7016.$$

The second equation from above will lead to the same solution for the ratio of a_2 to a_1 leads to the same answer. The two equations are not linearly independent. It is not possible to find unique solutions to both a_2 and a_1. The same method can now be used to find the second ratio of a_2 to a_1, by substitution of the second natural frequency into one of the two algebraic equations above. This leads to:

$\dfrac{a_2}{a_1} = \dfrac{-\omega_2^2 I_{z1} + k_{t1} + k_{t2}}{k_{t2}}$, where $\omega_2^2 = (165.5389)^2$, resulting in

$\dfrac{a_2}{a_1} = -2.7016$. Thus the two mode shapes may be written as vectors with each element normalized by a_1.

They are commonly written in a matrix of mode shapes as:

$$[U] = \begin{bmatrix} \begin{Bmatrix} a_1/a_1 \\ a_2/a_1 \end{Bmatrix}^{(1)} & \begin{Bmatrix} a_1/a_1 \\ a_2/a_1 \end{Bmatrix}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3.7016 & -2.7016 \end{bmatrix}$$

The superscripts on the vectors refer to the mode number, so the vector on the left is the mode shape of mode 1 and the one on the right is mode two. The physical meaning of each mode shape is as follows. Take mode 1. The mode shape means that if a_1=1 then a_2/a_1= a_2= 3.7016. Both are positive, meaning that both move in the same direction. a_1 represents the rotation of rotor number 1 and a_2 is the rotation of the second rotor. In free vibration in mode one, both rotors move in the same direction with the second rotor moving through and angle 3.7016 times greater than the first rotor. The frequency of this free vibration is at ω_1. A similar interpretation is true for motion in mode 2. However, the minus sign in the mode shape for a_2/a_1= -2.7016 means that the second rotor moves in the opposite direction to the first and with a magnitude 2.7016 times greater that the rotation of the first rotor.

Any undamped free(unforced) vibration of this two rotor system may be expressed as a weighted, linear combination of these two modal motions. That is to say,

$$\begin{Bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ \dfrac{a_2}{a_1} \end{Bmatrix}^{(1)} \cos(\omega_1 t - \varphi_1) + C_2 \begin{Bmatrix} 1 \\ \dfrac{a_2}{a_1} \end{Bmatrix}^{(2)} \cos(\omega_2 t - \varphi_2)$$

C_1 and C_2 are the modal weighting factors, and φ_1 and φ_2 are phase angles. Both depend on the initial conditions specified in the problem.

The solution for the natural frequencies and mode shape is usually done numerically, such as by using an eigenvalue solver in the program Matlab.

Solution to Problem 2, Problem set 11

a. The normalized matrix of mode shapes is

$$[U] = \begin{bmatrix} 1.0000 & 1.000 \\ 1.2945 & -1.5749 \end{bmatrix}$$ Note: Use lots of significant figures.

The transpose of [U] is given by

$$[U]^T = \begin{bmatrix} 1.0000 & 1.2945 \\ 1.0000 & -1.5749 \end{bmatrix}$$

and the inverse of $[U]^{-1}$ is know from linear algebra.

$$[A] = \frac{}{\det(A)} \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$ where the determinant of [A]=det[A]=$[ad - bc]$

$$[A]^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$[U]^{-1} = \begin{bmatrix}\begin{bmatrix}0.5488 & 0.3485 \\ 0.5412 & -0.3485\end{bmatrix}\end{bmatrix}$$

b. To find the modal mass, stiffness and damping matrices one must compute U^TMU, U^TKU, and U^TCU. Begin by stating for completeness the original mass, stiffness and damping matrices.

$$M = \begin{bmatrix} 1.5000 & 0.5000 \\ 0.5000 & 0.6670 \end{bmatrix}$$

$$K = \begin{bmatrix} 10 & 0 \\ 0 & 4.9050 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.025 \end{bmatrix}$$

Then compute the modal mass, damping and stiffness matrices.

$$M_N = \text{Modal mass} = U^T MU = \begin{bmatrix} 3.9123 & 0 \\ 0 & 1.5794 \end{bmatrix}$$

$$K_N = \text{Modal stiffness} = U^T KU = \begin{bmatrix} 18.2200 & 0 \\ 0 & 22.1654 \end{bmatrix}$$

$$C_N = \text{Modal damping} = U^T CU = \begin{bmatrix} 0.0919 & -0.0010 \\ -0.0010 & 0.1120 \end{bmatrix}$$

As expected, the modal mass and stiffness matrices are diagonal. The damping matrix is not quite diagonal. The off diagonal terms are small compared to the diagonal terms and in this case will be neglected. When the modal damping matrix has non-negligible off diagonal terms, there may be some coupling between modes, which is a subject best left to a more advanced course in vibration analysis. For the purposes of this problem, we shall ignore the off diagonal terms in the modal damping matrix. In other words, we set them to zero.

Hence $C_N = \text{Modal damping} = U^T CU = \begin{bmatrix} 0.0919 & 0.0 \\ 0.0 & 0.1120 \end{bmatrix}$

To check that the modal masses and stiffnesses are correct it should be possible to show that the natural frequencies may be retrieved directly from

$$\omega_1 = \sqrt{\frac{K_N(1,1)}{M_N(1,1)}} = \sqrt{\frac{18.22}{3.9123}} = 2.158\frac{radians}{second}$$

$$\omega_2 = \sqrt{\frac{K_N(1,1)}{M_N(1,1)}} = \sqrt{\frac{22.1654}{1.5794}} = 3.746\frac{radians}{second}$$

The individual modal damping ratios may also be computed directly from the modal mass and damping matrices as follows:

$$\zeta_1 = \frac{R_N(1,1)}{2\omega_1 M_N(1,1)} = \frac{0.0919}{2(2.148)(3.9123)} = 0.0055$$

$$\zeta_2 = \frac{R_N(2,2)}{2\omega_2 M_N(2,2)} = \frac{0.1120}{2(3.746)(1.5794)} = 0.0095$$

The beauty of modal analysis is that the equation of motion for each mode is a simple single degree of freedom oscillator such that we have two simple independent EOMs as shown below.

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

Where the relationship between the modal coordinates q_1 and q_2 and the generalized coordinates is given by:

$\begin{Bmatrix} x \\ \vartheta \end{Bmatrix} = U\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$ and the relationship between modal forces Q_1 and Q_2 and the generalized forces F(t) and $\tau(t)$ is given by $\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = U^T\begin{Bmatrix} F(t) \\ \tau(t) \end{Bmatrix}$

c. In this part of the problem there are no external generalized exciting forces, but there are initial conditions given as:

$\begin{Bmatrix} x_o \\ \vartheta_o \end{Bmatrix} = \begin{Bmatrix} 0.0 \\ 0.2rad \end{Bmatrix} = U\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix}$, where the second expression on the right comes from the definition of modal coordinates. Solving for $\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix}$ is accomplished by multiplying by U^{-1}.

$$\therefore U^{-1}U\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} = \begin{Bmatrix} q_{10} \\ q_{20} \end{Bmatrix} = U^{-1}\begin{Bmatrix} x_o \\ \vartheta_o \end{Bmatrix} = U^{-1}\begin{Bmatrix} 0 \\ 0.2 \end{Bmatrix} = \begin{bmatrix} 0.5488 & 0.3485 \\ 0.5412 & -0.3485 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.2 \end{Bmatrix} = \begin{Bmatrix} 0.0697 \\ -0.0697 \end{Bmatrix}$$

The initial velocities are zero. Hence $\dot{q}_{10} = \dot{q}_{20} = 0$. One may write directly from the initial conditions the transient modal response of each mode.

$$q_1(t) = q_{10}e^{-\zeta_1\omega_1 t}\cos(\omega_{1d}t) + \dot{q}_{10}e^{-\zeta_1\omega_1 t}\sin(\omega_{1d}t)$$
$$q_1(t) = 0.0697e^{-\zeta_1\omega_1 t}\cos(\omega_{1d}t)$$

$$q_2(t) = q_{20}e^{-\zeta_2\omega_2 t}\cos(\omega_{2d}t) + \dot{q}_{20}e^{-\zeta 2\omega 2t}\sin(\omega_{2d}t)$$
$$q_2(t) = -0.0697e^{-\zeta 2\omega 2t}\cos(\omega_{2d}t)$$

The response in the original generalized coordinates is given from the definition of the modal coordinates:

$$\begin{Bmatrix} x(t) \\ \vartheta(t) \end{Bmatrix} = U\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0 \\ 1.2945 & -1.5749 \end{bmatrix}\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$
$$\Rightarrow \vartheta(t) = 1.2945q_1(t)\text{-}1.5749q_2(t)$$
$$\vartheta(t) = 0.0902e^{-\zeta_1\omega_1 t}\cos(\omega_{1d}t) + 0.1098e^{-\zeta_2\omega_2 t}\cos(\omega_{2d}t)$$

Where $q_1(t)$ and $q_2(t)$ are taken from the expressions above. The result is a linear combination of the response of each mode weighted by its mode shape.

d. In the final part of the problem an external steady state excitation is given as

$$\begin{Bmatrix} F(t) \\ \tau(t) \end{Bmatrix} = \begin{Bmatrix} F_o \\ 0 \end{Bmatrix}\cos(\omega_2 t),$$ which may be used to find the modal excitations as follows,
$$\Rightarrow \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} = U^T\begin{Bmatrix} F(t) \\ \tau(t) \end{Bmatrix} = \begin{bmatrix} 1.0000 & 1.2945 \\ 1.0000 & -1.5749 \end{bmatrix}\begin{Bmatrix} F_o \\ 0 \end{Bmatrix}\cos(\omega_2 t) = \begin{Bmatrix} F_o \\ F_o \end{Bmatrix}\cos(\omega_2 t)$$

To find the steady state response of each mode requires the application of the steady state transfer function corresponding to the steady state solution of the equation of motion for each mode.

$$q_1(t) = \left| H_{q_1/Q_1}(\omega) \right| \left| Q_1 \right| \cos(\omega t - \varphi_1)$$

which when evaluated at $\omega = \omega_2$ yields

$$q_1(t) = \frac{F_o}{K_1} \frac{1}{\left[\left(1 - \frac{\omega_2^2}{\omega_1^2}\right)^2 + \left(2\zeta_1 \frac{\omega_2}{\omega_1}\right)^2 \right]^{1/2}} \cos(\omega_2 t - \varphi_1) =$$

$$q_1(t) = \frac{F_o}{K_1} \frac{1}{\left[(1 - 3.014)^2 + (0.0191)^2 \right]^{1/2}} \cos(\omega_2 t - \varphi_1) = 0.496 \frac{F_o}{K_1} \cos(\omega_2 t - \varphi_1)$$

$$\varphi_1 = -\tan^{-1}\left(\frac{2\zeta_1 \frac{\omega_2}{\omega_1}}{1 - \frac{\omega_2^2}{\omega_1^2}} \right) \bullet \pi$$

$$q_1(t) = 0.496 \frac{F_o}{K_1} \cos(\omega_2 t - \pi) = 0.0272 F_o \cos(\omega_2 t - \pi)$$

$$q_2(t) = \left| H_{q_2/Q_2}(\omega) \right| \left| Q_2 \right| \cos(\omega t - \varphi_2)$$

which when evaluated at $\omega = \omega_2$ yields

$$q_2(t) = \frac{F_o}{K_2} \frac{1}{\left[\left(1 - \frac{\omega_2^2}{\omega_2^2}\right)^2 + \left(2\zeta_2 \frac{\omega_2}{\omega_2}\right)^2 \right]^{1/2}} \cos(\omega t - \varphi_2) = \frac{F_o}{K_2} \frac{1}{2\zeta_2} \cos(\omega_2 t - \pi / 2)$$

$$\varphi_2 = -\tan^{-1}\left(\frac{2\zeta_2}{0} \right) = \pi / 2$$

$$q_2(t) = 2.373 F_o \cos(\omega_2 t - \pi / 2)$$

The contribution to the total response from mode 2 is two orders of magnitude greater than that due to mode one. This is because the system is being driven at the natural frequency of mode 2.

The mode superposition theorem allows us to write the system response in terms of the original generalized coordinates

$$\begin{Bmatrix} x(t) \\ \vartheta(t) \end{Bmatrix} = U \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{bmatrix} 1.0 & 1.0 \\ 1.2945 & -1.5749 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$

$\Rightarrow$

$$x(t) = q_1(t) + q_2(t)$$

$$x(t) = 0.0272F_o\cos(\omega_2 t - \pi) + 2.373F_o\cos(\omega_2 t - \pi/2)$$

$$\vartheta(t) = 1.2945q_1(t)\text{-}1.5749q_2(t)$$

$$\vartheta(t) = 0.0352F_o\cos(\omega_2 t - \pi) + 3.737F_o\cos(\omega_2 t - \pi/2)$$

Mode 2 dominates the response because the system is being driven at the natural frequency of mode 2.

Solution to Problem 3
Problem set 11

This problem is a simplification of Problem 2 above in which the original non-linear equations of motion were found to be:

$$\left(m_1 + m_2\right)\ddot{x} + b\dot{x} + kx + m_2\frac{l}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0 \qquad (1)$$

$$m_2\ddot{x}\frac{l}{2}\cos\theta + m_2\frac{l^2}{3}\ddot{\theta} + m_2 g\frac{l}{2}\sin\theta + c\dot{\theta} = 0, \text{ and} \qquad (2)$$

In this problem the rotation is at a known constant rate, *ω*.

$\theta(t) = \Omega t,\ \dot{\theta} = \Omega$, and $\ddot{\theta} = 0$. Substitution into Equation (1) above leads to:

$$\left(m_1 + m_2\right)\ddot{x} + b\dot{x} + kx = m_2\frac{l}{2}\omega^2\sin\Omega t = F_o\sin\Omega t \qquad (3)$$

Equation (2) is not needed. It came from summing the torques about the pivot point. Now that the arm is travelling at constant rotation rate, there would need to be an additional external torque applied at the pivot, so as to make the arm travel at a constant rate. Equation (2) could be used to find that torque if it were needed. It will be ignored in this problem.

The equation of motion for the cart in the x direction comes from Equation (3). The rotating arm is the source of an harmonic force in the x direction given by

$$F_o(t) = m_2\frac{l}{2}\Omega^2\sin\Omega t$$

where $\frac{l}{2} = 0.01m$ and $m_2 = 0.1kg$.

$$\Rightarrow F_o(t) = 0.001\Omega^2\sin\Omega t$$

Equation (3) is the equation of a single degree of freedom, mass-spring-dashpot oscillator with a total mass of m_1+m_2. The ratio of the magnitude of the force transmitted to the wall through the spring and dashpot to the input force F_o is known from our study of vibration isolation, and is given by:

$$\left|\frac{F_T}{F_o}\right| = \frac{\left[1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}}{\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}}, \text{ where } \omega_n = \sqrt{\frac{K}{m_1+m_2}} = \sqrt{\frac{10}{1.1}} = 3.015 \text{ radians/s,}$$

$$\zeta = \frac{b}{2(m_1+m_2)\omega_n} = \frac{.05}{2(1.1)3.015} = 0.0075$$

You are asked to evaluate this at three values of the frequency ratio $\frac{\Omega}{\omega_n} = 0.1, 1.0, \text{ and } 5.0$. The results for these three frequency ratios in the same order are:

$$\frac{|F_T|}{|F_o|} = 1.01, \quad 66.7, \text{ and } 0.0418$$

Solution to Problem 4
Problem set 11

The key to estimating the effect of the mass of the spring on the natural frequency is to take into account the kinetic energy of the spring itself. It is assumed that the spring motion is quasi-static. That is the spring does not exhibit any dynamic motion other that following the motion of the cart. In other words, the natural frequencies of the spring as a flexible continuous system are much higher than the frequency of oscillation in this problem.

The position of the point of attachment of the spring to the cart is given by the coordinate x, where x is measured from an origin which coincides with the zero spring force position of the cart.

The spring has zero velocity at its point of attachment to the non-moving wall and moves with the velocity of the cart at the point of attachment to the cart. At all other points on the spring the velocity is linearly proportional to the distance from the wall. The spring has mass per unit length ‘m’ and total mass ml_o.

We are interested in the maximum kinetic energy of the spring. This occurs when the spring has zero displacement, x, and the spring has zero potential energy. In order to compute the kinetic energy of the spring we need to know the velocity of every point on the spring as x(t) passes through the point x=0. The displacement and velocity of the cart in free vibration are given by:

$$x(t) = x_o \sin(\omega_n t)$$
$$\dot{x} = x_o \omega_n \cos(\omega_n t)$$

At the moment x=0, the velocity of every point on the spring is given by:

$\dot{s} = x_o \omega_n \dfrac{s}{l_o}$, where s is a dummy coordinate which is measured from the point of attachment to the wall to the point of attachment to the cart, which is assumed to be at x=0, the unstretched spring position.

The total maximum kinetic energy of the system must equal the total maximum potential energy that is stored in the spring, when the cart is at zero velocity and maximum displace, $x=x_o$.

The maximum KE is given by the sum of the KE of the cart plus that of the spring.

$$KE_{total} = KE_{spring} + KE_{cart} = PE = \frac{1}{2} k x_o^2$$
$$KE_{total} = \frac{1}{2} M (x_o \omega)^2 + \int_0^{l_o} \frac{1}{2} m \left(\frac{s}{l_o} x_o \omega \right)^2 ds = \frac{1}{2} M (x_o \omega)^2 + \frac{1}{2} \frac{m l_o}{3} (x_o \omega)^2$$
$$KE_{total} = \frac{1}{2} \left(M + \frac{m l_o}{3} \right) (x_o \omega)^2 = PE_{\max} = \frac{1}{2} k x_o^2$$

Solving for $\omega^2 = \dfrac{k}{M + \dfrac{m l_o}{3}}$ leads to the result that the natural frequency of the system including the mass of the spring is given by:

$$\omega_n = \sqrt{\frac{k}{M + \dfrac{m l_o}{3}}}$$

The effective mass of the spring is one third of its total mass. The equivalent equation of motion is given by:

$$\left(M + \frac{m l_o}{3} \right) \ddot{x} + kx = 0.$$

b. The ratio $\frac{\omega_n}{\omega_{no}} = \sqrt{\frac{M}{M + \frac{ml_o}{3}}}$