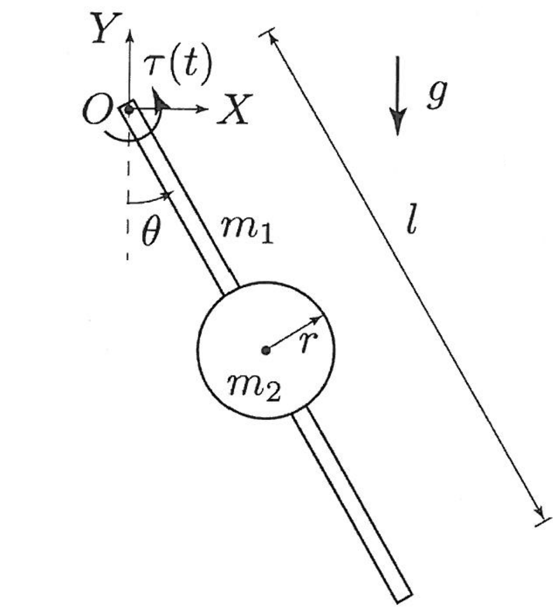
2.003SC Engineering Dynamics
Quiz 2

Problem 1 (25 pts)

A cuckoo clock pendulum consists of two pieces glued together:

- a slender rod of mass m_1 and length l, and
- a circular disk of mass m_2 and radius r, centered at the slender rod's midpoint.

The pendulum is attached at one end to a fixed pivot, O, as shown below, where a time-varying torque, $\tau(t)\hat{K}$, is applied as well. Note that gravity acts.



a) *(8pts)* Find the expression for the pendulum's mass *moment of inertia* I_{zz} *about* O.

b) *(4pts)* Find an expression for the pendulum's *angular momentum about* O.

c) *(5pts)* Draw a *free body diagram* for the system.

d) *(8pts)* Find the equation(s) of motion of the pendulum *by the direct method.*

Solution:

a) **Moment of Inertia**
The moments of inertia of the rod and the disk about their centers of mass are, respectively:

$$I_G^{rod} = \frac{1}{12}m_1 l^2 \qquad I_G^{disk} = \frac{1}{2}m_2 r^2$$

From the parallel axis theorem, the moment of inertia of the entire pendulum about point O is:

$$I_O = \left[m_1\left(\frac{l}{2}\right)^2 + \frac{1}{12}m_1 l^2\right] + \left[m_2\left(\frac{l}{2}\right)^2 + \frac{1}{2}m_2 r^2\right]$$

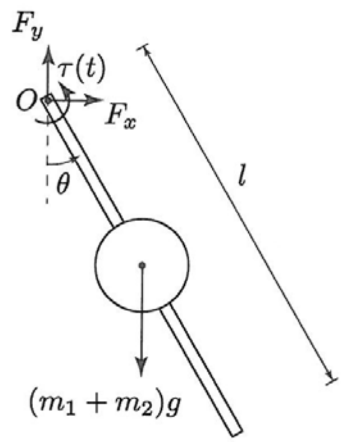
or

$$I_O = \frac{1}{3}m_1 l^2 + m_2\left(\frac{r^2}{2} + \frac{l^2}{4}\right)$$

b) **Angular Momentum**
The angular momentum about point O for the entire pendulum is:

$$\vec{H}_O = I_O\omega = I_O\dot{\theta}\hat{k}$$

c) **Free Body Diagram**
The free body diagram for the system contains the weight, the reaction forces and the torque:



d) **Equation of Motion**
Because the pivot is fixed, we can sum torques about point O and use the following:

$$\sum \vec{\tau}_O = \frac{d\vec{H}_O}{dt}$$

$$\tau(t) - (m_1 + m_2)g\frac{l}{2}\sin\theta = I_0\ddot{\theta}$$

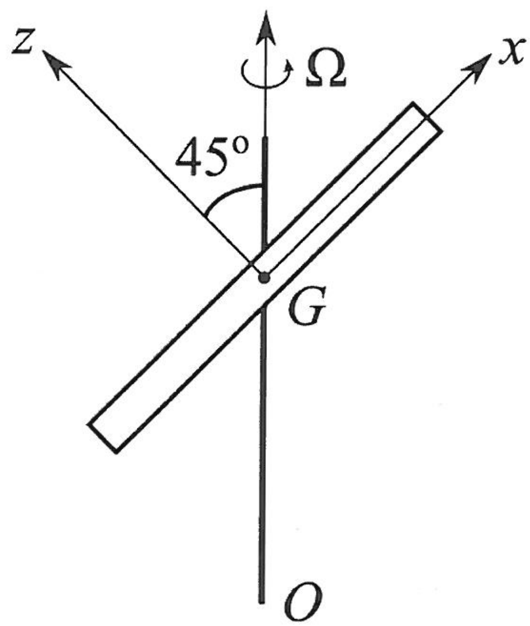
or

$$\left[\frac{1}{3}m_1 l^2 + m_2\left(\frac{r^2}{2} + \frac{l^2}{4}\right)\right]\ddot{\theta} + (m_1 + m_2)g\frac{l}{2}sin\theta = \tau(t)$$

Problem 2 (25 pts)

A thin disk rotates about an axis which passes through the center of mass of the disk. The disk is inclined at 45° angle with respect to the axis of rotation as shown in the figure. G_{xyz} are body fixed principal axes and the inertia matrix for the disk is given as

$$[I_G] = \begin{bmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix} \text{ in the } G_{xyz} \text{ body fixed coordinates.}$$



a) *(9pts) Find the angular momentum of the system with respect to the* G, the center of mas of the disk. Express your answer in terms of the three vector components: $\vec{H} = H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$.

b) *(10pts)* Find the *torque, which must be applied at G to cause this disk* to rotate as shown in the figure. Do not assume that the rotation rate Ω is constant.

c) *(3pts)* Is this rotor *statically balanced*?

d) *(3pts)* Is this rotor *dynamically balanced*?

Solution:

We need $\vec{\omega}$ in body coordinates: $\vec{\omega} = \omega_x\hat{i} + O\hat{j} + \omega_z\hat{k}$.

a) $\vec{H}_G = I_G\vec{\omega} = I_{xx}\omega_x\hat{i} + Iyy\omega_y\hat{j} + I_{zz}\omega_z\hat{k}$
where

$$\vec{\omega} = \frac{\sqrt{2}}{2}\Omega\hat{i} + \frac{\sqrt{2}}{2}\Omega\hat{k} + O\hat{j}$$
$$= \omega_x\hat{i} + \omega_z\hat{k}, \text{ expressed in components in the rotating frame } G_{xyz},\ \omega_y\hat{j} = 0.$$

b)

$$\vec{\tau}_G = \frac{d\vec{H}_G}{dt} = \left(\frac{d\vec{H}_G}{dt}\right)_{G_{xyz}} + \vec{\omega} \times \vec{H}_G$$

$$= I_{xx}\dot{\omega}_x i + I_{zz}\dot{\omega}_z k + (\omega_x i + \omega_z k) \times (I_{xx}\omega_x i + I_{zz}\omega_z k)$$

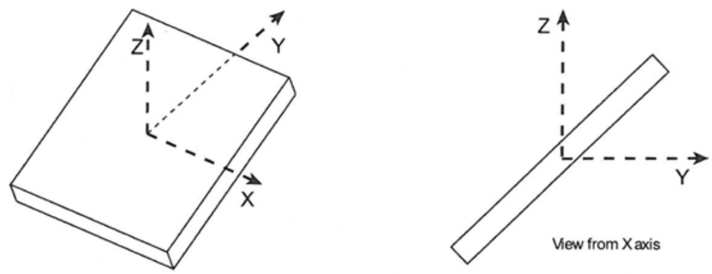
$$\tau_{ext} = I_{xx}\dot{\omega}_x i + I_{zz}\dot{\omega}_z k - I_{zz}\omega_x\omega_z j + I_{xx}\omega_x\omega_z j$$

c) (and d) It is statically balanced but not dynamically balanced.

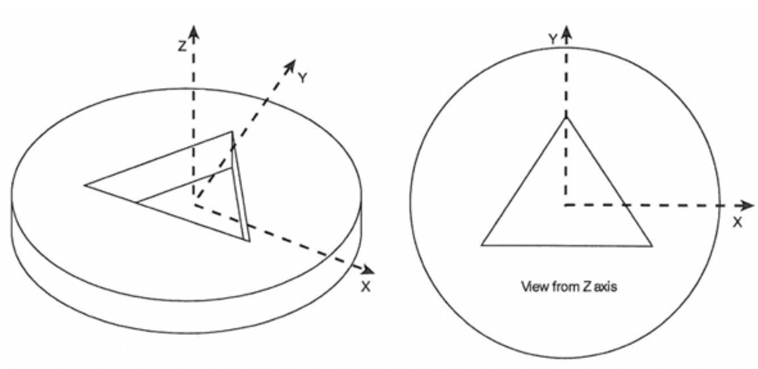
Problem 3 (9 pts)

For each of the following uniform density objects, determine whether the set of axes depicted are a set of *principal axes.* Note that two views are provided for each object.

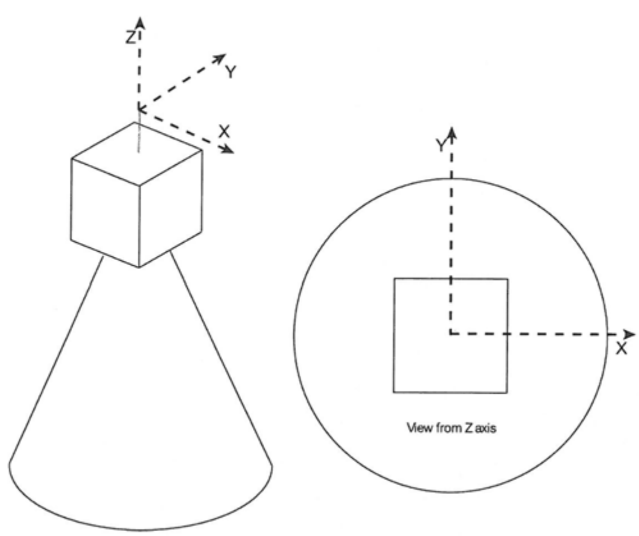
a) *(3 pts)* The axis shown are principal axes: TRUE or FALSE



b) *(3 pts)* In this object the triangular cutout is an equilateral triangle, centered in the disk.
The axis shown are principal axis: TRUE or FALSE



c) The axis shown are principal axes: TRUE or FALSE



Solution:

a) False

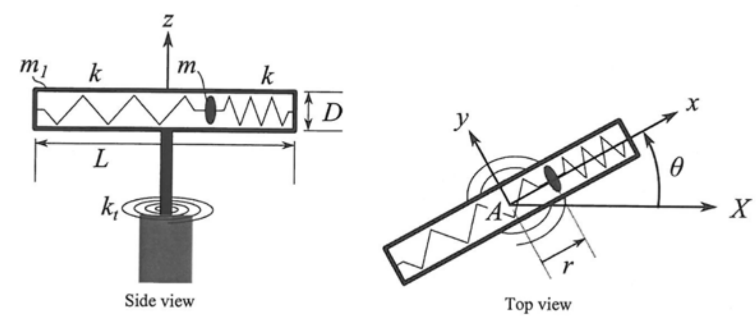
b) True

c) True

Problem 4

A massless vertical rod is welded to a horizontal slender tube of length L, diameter D and mass m_l. the vertical rod is supported in a frictionless bearing. Attached to the vertical rod is a torsional spring with spring constant k_t with units of Nm/rad. A mass m is attached to both ends of the tube, with two springs, each of spring constants is k, and unstretched length is $L/2$. The mass slides frictionlessly in the tube. The mass may be treated as a point mass. The inertia matrix for the tube expressed in its body fixed axes A_{xyz} is given approximately by:

$$[I] = \begin{bmatrix} \frac{1}{4}m_1 D^2 & 0 & 0 \\ 0 & \frac{1}{12}m_1 L^2 & 0 \\ 0 & 0 & \frac{1}{12}m_1 L^2 \end{bmatrix}$$



(θ, r) defined on the top view of the figure form a set of complete and generalized coordinates, which describe this two degree of freedom system. You should not assume that (θ, r) or their first two time derivatives are zero.

a) *(15pts)* Calculate the *kinetic energy T, potential energy V, and generalized forces* Q_r and Q_θ for this system.

b) *(10pts)* Derive the *equations of motion* for the system *using Lagrange* method and using the generalized coordinates (θ, r).

Solution:

a)

$$T = T_{\text{rod}} + T_{\text{tube}} + T_{\text{mass}}$$

$$T_{\text{rod}} = 0 \text{ because } I_{\text{rod}} = 0$$

$$T_{\text{tube}} = \frac{1}{2}\vec{\omega}^T [I] \vec{\omega} = \frac{1}{2}\begin{bmatrix} 0 & 0 & \omega_z \hat{k} \end{bmatrix} [I] \begin{Bmatrix} 0 \\ 0 \\ \omega_z \hat{k} \end{Bmatrix}$$

$$= \frac{1}{2} I_{zz}\omega_2^2 = \frac{1}{2} I_{zz}\dot{\theta}^2 = \frac{1}{24} m_1 L^2 \dot{\theta}^2$$

$$T_{\text{mass}} = \frac{1}{2} m \vec{v}_p \cdot \vec{v}_p, \text{ where } \vec{v}_p = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$= \frac{1}{2} m \left(\dot{r} + r^2 \dot{\theta}^2\right)$$

$$V = \frac{1}{2} k_T \theta^2 + 2 \quad \frac{1}{2} k r^2$$

$Q_r = Q_\theta = 0$ because there are no nonconservative forces.

b) **Equation of motion in θ direction**

$$
\begin{aligned}
\frac{d}{dt}\quad\frac{\partial T}{\partial \dot{\theta}}\quad - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} &= Q_\theta = 0 \\
\frac{\partial T}{\partial \theta} &= 0 \\
\frac{d}{dt}\quad\frac{\partial T}{\partial \dot{\theta}}\quad = \quad \frac{m_1 L^2}{12} + mr^2 \quad \ddot{\theta} &+ 2mr\dot{r}\dot{\theta} \\
\frac{\partial V}{\partial \theta} = k_T\theta \Rightarrow \textbf{The } \theta \textbf{ EOM is given by} & \\
\frac{m_1 L^2}{12} + mr^2 \quad \ddot{\theta} + 2mr\dot{r}\dot{\theta} + k_T\theta &= 0
\end{aligned}
$$

EOM in r direction

$$
\begin{aligned}
\frac{d}{dt}\quad\frac{\partial T}{\partial \dot{r}}\quad - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} &= Q_r \\
\frac{d}{dt}\quad\frac{\partial T}{\partial \dot{r}}\quad &= m\ddot{r} \\
-\frac{\partial T}{\partial r} &= -mr\dot{\theta}^2 \\
+\frac{\partial V}{\partial r} &= 2kr
\end{aligned}
$$

$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + 2kr = 0$ **is the r equation of motion.**

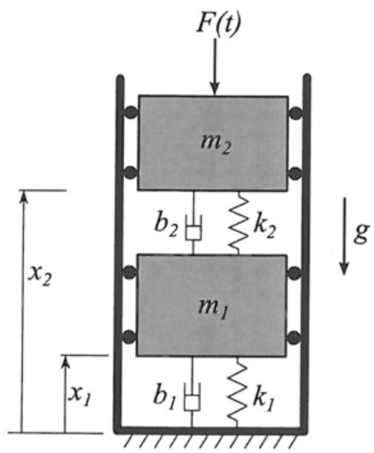
Problem 5

Two blocks of mass m_1 and m_2 are frictionlessly constrained to vertical motion. The first block is connected to the ground via a spring and a dashpot with constants k_1 and b_1 as shown. the second block is connected to the first one via a spring and a dashpot with constants k_2 and b_2. A force $F(t)$ is applied to the second mass as shown of the figure.

The vertical position of the first block is denoted by x_1 while the position of the second one is denoted by x_2. (x_1, x_2) form a set of complete and independent generalized coordinates to describe this two degrees of freedom system.

a) *(8pts)* Calculate the generalized force Q_1 associated with the generalized displacement x_1.

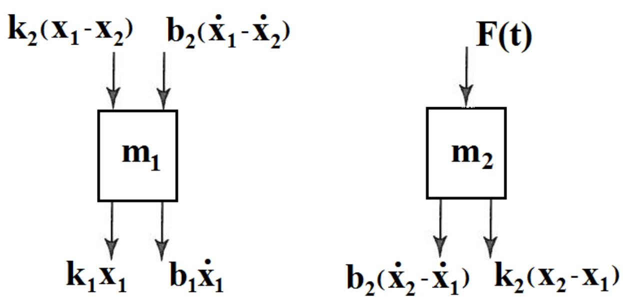
b) *(8pts)* Calculate the generalized force Q_2 associated with the generalized displacement x_2.



Solution:

Free body diagrams:

$$Q_1 = -\,b_1\dot{x}_1 - b_2(\dot{x}_1 - \dot{x}_2)$$
$$Q_2 = -\,b_2(\dot{x}_2 - \dot{x}_1) - F(t)$$



2.003SC / 1.053J Engineering Dynamics
Fall 2011