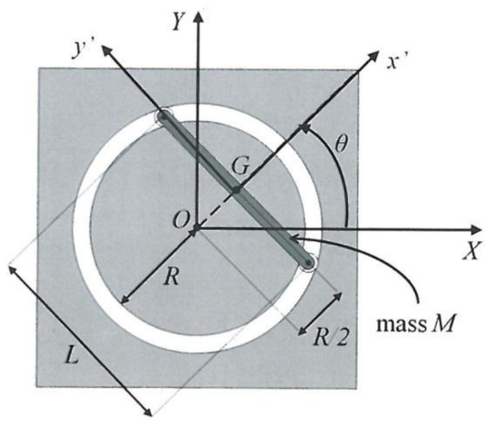
2.003 Engineering Dynamics
Problem Set 8--Solution

This was an optional homework handed out as a study aid in advance of the second quiz in the course in the fall of 2011. It is an actual quiz from a prior year. There were no concept questions. Students had 90 minutes to work the quiz and were allowed two pages of notes, both sides of the page. No textbooks allowed.

Problem 1:

A slender rod of mass M and length, L, moves around a circular track under the influence of gravity. The small rollers at the ends of the rod are confined to the circular track and roll without friction. Neglect the mass of the rollers. The radius of the track is R and $L=\sqrt{3}R$. Find an equation of motion of the rod as a function of the coordinate, θ, as shown in the figure. The frame $G_{x''y''z''}$ is fixed to the center of mass of the rod.



Solution: This is a planar motion problem in which a rigid body rotates about a fixed point at 'O'. For review purposes begin with the full 3D vector equation expressing Euler"s law for rotation of rigid bodies.

$$\sum_i \boldsymbol{\tau}_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} + \vec{\mathbf{v}}_{A/O} \times \vec{\mathbf{P}}_{G/O}$$

Because this is rotation about a fixed point the 2nd term on the right is zero. Because this is a planar motion problem about a fixed point at 'O'='A', the angular momentum with respect to the fixed point may be expressed using the parallel axis theorem and the moment of inertia with respect to the center of mass of the rod. Note that the body-fixed frame, $G_{x'y'z,}$ has axes which are principal axes.

$$\vec{H}_{/A} = \left[I_{/A}\right]\{\omega\} = \left[I_{/G} + Md^2\right]\omega_z\hat{k} = \left[M\frac{L^2}{12} + Md^2\right]\omega_z\hat{k}, \text{ where A=O, d=}\frac{R}{2} \text{ and L=}\sqrt{3}R.$$

$$\vec{H}_{/O} = M\frac{R^2}{2}\omega_z\hat{k} = M\frac{R^2}{2}\dot{\theta}\hat{k}$$

$$\frac{d\vec{H}_{/O}}{dt} = M\frac{R^2}{2}\ddot{\theta}\hat{k}$$

The torque that results from the gravitational force acting on the body at the center of mass is given by

$$\sum_i \boldsymbol{\tau}_{i/O} = \vec{r}_{G/O} \times -Mg\hat{J} = -Mg\frac{R}{2}\cos\boldsymbol{\theta}\hat{k}$$

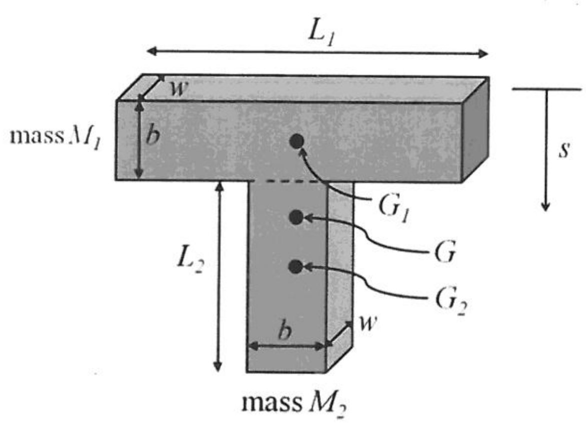
Equating the external torque with the time rate of change of the angular momentum leads to the equation of motion :

$$\sum_i \boldsymbol{\tau}_{i/O} = \frac{d\vec{H}_{/O}}{dt} = M\frac{R^2}{2}\ddot{\boldsymbol{\theta}}\hat{k} = -Mg\frac{R}{2}\cos\boldsymbol{\theta}\hat{k}$$

$$\Rightarrow R\ddot{\boldsymbol{\theta}} + g\cos\boldsymbol{\theta} = 0$$

Problem 2:

A T-shaped object is made from two rectangular solid blocks of uniform density. Both blocks have thickness "w" and width "b". The horizontal part of the "T" has length L_1 and mass M_1. The vertical part has length L_2 and mass M_2. G_1 and G_2 are the centers of mass of each block and are located at the geometric centers of each block.



a). Find the location of the center of mass of the combined object, which lies somewhere on the vertical line which passes through G_1 and G_2. Express it as a distance "s" measured from the top of the "T", as shown in the figure.

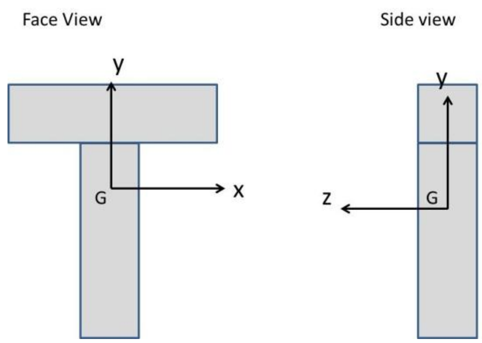
b). On a sketch of the combined object, draw the three principal axes, with their origin at "G" the center of mass of the combined object. Label them as an xyz coordinate system. It is most important that you show the orientation of the axes with respect to the object.

Solution: a). Let s_G be defined as the location of the center of mass in the s coordinate system. From the definition of center of mass, its location in the vertical direction may be computed by:

$$(M_1 + M_2)s_G = M_1 s_1 + M_2 s_2 = M_1\frac{b}{2} + M_2(b + \frac{L_2}{2})$$

$$\Rightarrow s_G = \frac{M_1\frac{b}{2} + M_2(b + \frac{L_2}{2})}{M_1 + M_2}$$

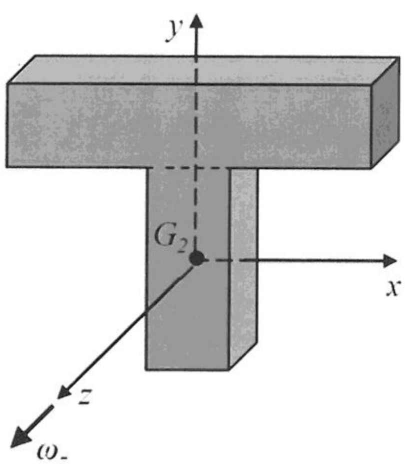
b). The principal are shown passing through G in the drawing below.



They must be principal axes, because they satisfy the following symmetry properties. The “z” axis is perpendicular to one plane of symmetry. The “x”axis is perpendicular to another plane of symmetry and the „y‟axis is perpendicular to x and z.

Problem 3:

The T-shaped object described in the previous problem is rotated at constant speed, ω_z , about the z-axis shown in the figure below. The axis passes through, G_2, the center of mass of the vertical block and is perpendicular to the plane of the „T‟.



a). Compute the force which must be applied to the axle to keep it from moving as the object rotates.

b). The presence of this force is evidence that the body has what type of imbalance? A. Static, B. Dynamic, C. Both.

Solution: Consider question b) first: The axis of rotation, z, passing through G_2 is a principal axis because it is parallel to the z principal axis found in question 2. Because the rotation is about a principal axis it will not produce a dynamic imbalance. However, because this axis of rotation does not pass through the center of mass of the combined object the rotation will produce a static imbalance. Answer ‘A’ is correct. Now we turn to computing the force on the axel.

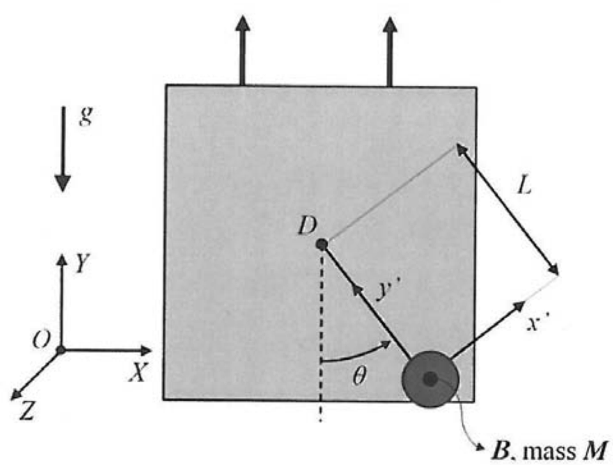
a). Force is required to produce a change in the linear momentum of a rigid body as specified in Newton’s 2nd law.

$\sum \vec{F}_{ext} = \dfrac{d\vec{P}_{/O}}{dt} = \left(M_1 + M_2\right)\vec{a}_{G/O}$. The only source of acceleration at G is due to the constant rotation about a fixed point, which produces a centripetal acceleration, which we know from many previous problems to be given by $\vec{a}_{G/O} = -e\omega_z^2\hat{j}$, where ‘e’ is the distance between G_2 and G and could be computed from the results in Problem 2. Therefore we can express the

external force on the rotating rigid body as $\vec{F}_{ext} = (M_1 + M_2)\vec{a}_{G/O} = -(M_1 + M_2)e\omega_z^2 \hat{j}$.
From Newton's 3rd law, the force on the axel must be equal and opposite. Therefore,
$\vec{F}_{axel} = -(M_1 + M_2)\vec{a}_{G/O} = (M_1 + M_2)e\omega_z^2 \hat{j}$ is the force the arm places on the axle. The force applied by the bearing on the axle is by Newton's 3rd law in the opposite direction.
$\vec{F}_{bearing} = -(M_1 + M_2)\vec{a}_{G/O} = (M_1 + M_2)e\omega_z^2 \hat{j}$

Problem 4:

A simple pendulum made of a point mass, M, suspended in an elevator from point D. O_{xyz} is a fixed inertial reference frame with unit vectors $\hat{I}, \hat{J}, \hat{K}$. The $B_{x'y'z'}$ frame is attached to the pendulum, as shown, with unit vectors $\hat{i}, \hat{j}, \hat{k}$. It is strongly suggested that you use these reference frames and unit vectors in solving the problem below.



The elevator has velocity $\dot{Y}$ and acceleration $\ddot{Y}$. Their values are given and therefore '*Y*' does not have to be considered an independent degree of freedom, because it is constrained to fixed values. The elevator is also not allowed to move in the *X* or *Z* directions. This is a planar motion problem, requiring one independent coordinate, θ, as shown in the figure to the left.

One method that can be used to obtain the equation of motion of this pendulum is to apply the physical law relating angular momentum to torque, about a moving point, "A", as given below. 'G' in the equation is the center of mass of the rigid body.

$$\sum_i \tau_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} + \vec{v}_{A/O} \times \vec{P}_{G/O} \qquad (1)$$

Equation (1) is true for all rigid bodies. A very useful variation for finding $\vec{H}_{/A}$ is given in Equation (2) below, which is also generally true for rigid bodies.

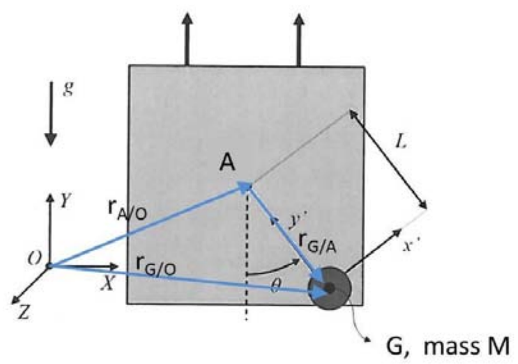
$$\vec{H}_{/A} = \vec{H}_{/G} + \vec{r}_{G/A} \times \vec{P}_{G/O} \qquad (2)$$

a) In a sketch define the rigid body in this problem. Mark clearly in your sketch which points correspond to the points 'A' and 'G', which are referred to in the equations above.

b) Find for this particular problem the vector expressions for $\vec{v}_{A/O}$, $\vec{P}_{G/O}$, $\sum_i \vec{\tau}_{i/A}$, and $\vec{H}_{/A}$.

c) Find the equation of motion. This takes time so save it to the end of the exam.

Solution:



a). In the figure above the point D is the point about which the angular momentum is to be computed. Hence D=A in equations (1) and (2). The point labeled 'B' in the figure is the center of mass for the rigid body and therefore B=G in equations (1) and (2). These are shown in the figure to the left. The rigid body in this case is the combination of the massless pendulum arm and the concentrated mass, M. Also shown in the figure are the position vectors which satisfy the relationship that $\vec{r}_{G/O} = \vec{r}_{A/O} + \vec{r}_{G/A}$.

b). You are to find: $\vec{v}_{A/O}$, $\vec{P}_{G/O}$, $\sum_i \vec{\tau}_{i/A}$, and $\vec{H}_{/A}$.

i. $\vec{v}_{A/O}$ is the velocity of the elevator, given as $\dot{Y}\hat{J}$.

ii. $\vec{P}_{G/O} = M\vec{v}_{G/O} = M\left(\vec{v}_{A/O} + \vec{v}_{G/A}\right) = M\left(\dot{Y}\hat{J} + \omega\hat{k} \times -L\hat{j}\right) = M\left(\dot{Y}\hat{J} + L\dot{\theta}\hat{i}\right)$

where we have found $\mathbf{v}_{G/A}$ from the equation for the derivative of the rotating vector $\mathbf{r}_{G/A}$.

$$\vec{v}_{G/A} = \left(\frac{d\vec{r}_{G/A}}{dt}\right)_{O_{xyz}} = \left(\frac{d\vec{r}_{G/A}}{dt}\right)_{B_{x'y'z'}} + \vec{\omega} \times \vec{r}_{G/A} = 0 + \omega_z\hat{k} \times -L\hat{j} = L\omega_z\hat{i} = L\dot{\theta}\hat{i}$$

The external torques applied to the rigid body with respect to point „A" come only from the gravitational force acting on the concentrated mass.

iii. $\vec{\tau}_{/A} = \vec{r}_{G/A} \times -Mg\hat{J} = -L\hat{j} \times -Mg\hat{J}$. But this expression has the cross product of unit vectors defined in different frames. One must be converted to the other. Note that $\hat{j} = -\sin\theta\hat{I} + \cos\theta\hat{J}$ and therefore $\tau_{/A} = -L(-\sin\theta\hat{I} + \cos\theta\hat{J}) \times -Mg\hat{J} = -MgL\sin\theta\hat{k}$

Take note that the unit vector in the z direction is the same for both the O_{XYZ} and $B_{x'y'z'}$ reference frames. $\hat{k}$ is used for both in this solution.

iv. To find $\vec{H}_{/A}$ it generally a good idea to use Equation (2), which provides that $\vec{H}_{/A} = \vec{H}_{/G} + \vec{r}_{G/A} \times \vec{P}_{G/O}$.

Note that $I_{zz/G}$=0 because M is a point mass and therefore $\vec{H}_{/G} = 0$.

$$\vec{H}_{/A} = \vec{H}_{/G} + \vec{r}_{G/A} \times \vec{P}_{G/O} = 0 + (-L\hat{j} \times M\left(\dot{Y}\hat{J} + L\dot{\boldsymbol{\theta}}\hat{i}\right) = -M\dot{Y}L\hat{j} \times \hat{J} - ML^2\dot{\boldsymbol{\theta}}\, j \times \hat{i}$$

Note that $\hat{j}$=-sin$\boldsymbol{\theta}\hat{I}$+cos$\boldsymbol{\theta}\hat{J}$, such that

$$\vec{H}_{/A} = (-M\dot{Y}L(-\sin\boldsymbol{\theta}\hat{I} + \cos\boldsymbol{\theta}\hat{J}) \times \hat{J} + ML^2\dot{\boldsymbol{\theta}})\hat{k} = M\dot{Y}L\sin\boldsymbol{\theta}\hat{k} + ML^2\dot{\boldsymbol{\theta}}\hat{k}$$

Now taking the time derivative of the angular momentum provides:

$$\frac{d}{dt}\vec{H}_{/A} = \frac{d}{dt}(M\dot{Y}L\sin\boldsymbol{\theta}\hat{k} + ML^2\dot{\boldsymbol{\theta}}\hat{k}) = M\dot{Y}L\dot{\boldsymbol{\theta}}\cos\boldsymbol{\theta}\hat{k} + M\ddot{Y}L\sin\boldsymbol{\theta}\hat{k} + ML^2\ddot{\boldsymbol{\theta}}\hat{k} \qquad (3)$$

c). The final question was to find the equation of motion. The equation available to accomplish this is Equation (1)

$$\sum_i \tau_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} + \vec{v}_{A/O} \times \vec{P}_{G/O}$$

We have evaluated the left side of this equation and the first term on the right hand side. All that remains is to evaluate the second term on the RHS

$$\begin{aligned}\vec{v}_{A/O} \times \vec{P}_{G/O} &= \dot{Y}\hat{J} \times M\left(\dot{Y}\hat{J} + L\dot{\theta}\hat{i}\right) = ML\dot{\theta}\dot{Y}(\hat{J} \times \hat{i}) = ML\dot{\theta}\dot{Y}(\cos\theta\hat{j} + \sin\theta\hat{i}) \times \hat{i} \\ \vec{v}_{A/O} \times \vec{P}_{G/O} &= -ML\dot{\theta}\dot{Y}\cos\theta\hat{k}\end{aligned} \qquad (4)$$

Substituting all the information into Equation(1) results in :

$$\sum_i \vec{\tau}_{i/A} = -MgL\sin\theta\hat{k} = (ML^2\ddot{\theta} + M\ddot{Y}\sin\theta)\hat{k} \qquad (5)$$

Upon rearranging Equation (5) the final EOM is found.

$$ML^2\ddot{\theta} + ML(\ddot{Y} + g)\sin\theta = 0 \qquad (6)$$

Hence, the acceleration of the elevator just acts to increase the effective value of the acceleration of gravity. This pendulum would reveal an increase in its natural frequency.

That‵s the end to the problem as specified. However, it was pretty tedious evaluating all those terms in equation (1). Is it all necessary? Close inspection of the last step which led to equation (5) reveals that the term $\vec{v}_{A/O} \times \vec{P}_{G/O}$ was simply cancelled out by a term produced by the computation of $\left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}}$. After doing many such problems, as the lecturer in charge of the course, I concluded that there must be a simpler way to do this that will apply in all cases. This result is presented in the following brief appendix. It is a valuable result, not covered elsewhere in this course an rarely included in textbooks on dynamics.

Appendix: The simpler way to compute $\sum_i \tau_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} + \vec{v}_{A/O} \times \vec{P}_{G/O}$ (1)

An alternative and far less tedious option is:

$$\sum_i \vec{\tau}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O} \qquad (7)$$

Here is the derivation:

There are two terms on the right hand side of Equation (1) 1. $\left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}}$ and 2. $\vec{v}_{A/O} \times \vec{P}_{G/O}$

Note that the use of the subscript O_{XYZ} in the expression for the time derivative $\left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}}$

means the derivative must be taken with respect to the inertial reference frame.

To show that the $\vec{v}_{A/O} \times \vec{P}_{G/O}$ term will cancel out we begin by replacing $\vec{H}_{/A}$ with $\vec{H}_{/A} = \vec{H}_{/G} + \vec{r}_{G/A} \times \vec{P}_{G/O}$ and taking the derivative $\left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}}$.

1. $$\left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} = \left(\frac{d\vec{H}_{/G}}{dt}\right)_{O_{xyz}} + \vec{r}_{G/A} \times M\vec{a}_{G/O} + \vec{v}_{G/A} \times \vec{P}_{G/O}$$

Note that the key substitution in the proof is $\vec{v}_{G/A} = \vec{v}_{G/O} - \vec{v}_{A/O}$

$$\therefore \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} = \left(\frac{d\vec{H}_{/G}}{dt}\right)_{O_{xyz}} + \vec{r}_{G/A} \times M\vec{a}_{G/O} + \left(\vec{v}_{G/O} - \vec{v}_{A/O}\right) \times \vec{P}_{G/O}$$

and note that $\vec{v}_{G/O} \times \vec{P}_{G/O} = 0,$ always.

$$\therefore \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} = \left(\frac{d\vec{H}_{/G}}{dt}\right)_{O_{xyz}} + \vec{r}_{G/A} \times M\vec{a}_{G/O} - \vec{v}_{A/O} \times \vec{P}_{G/O}$$

The last bit in the last line is $-\vec{v}_{A/O} \times \vec{P}_{G/O}$ which will cancels the second term in equation (1) leaving us with:

$$\sum_i \vec{\tau}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O} \qquad (7)$$

which is equivalent to Equation (1)

$$\sum_i \vec{\tau}_{i/A} = \left(\frac{d\vec{H}_{/A}}{dt}\right)_{O_{xyz}} + \vec{v}_{A/O} \times \vec{P}_{G/O} \qquad (1)$$

Equations (1) and (7) are equivalent, but (7) requires much less work to obtain.

Applying it to problem 4 above results in:

$$\sum_i \vec{\tau}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O} \qquad (3)$$

$\sum_i \vec{\tau}_{i/A} = -MgL\sin\theta\hat{k},$ from part b).

$\frac{d\vec{H}_{/G}}{dt} = 0,$ because M is a point mass.

$$\vec{a}_{G/O} = \left(\frac{d\vec{v}_{G/O}}{dt}\right)_{O_{XYZ}} = \left(\frac{d\left(\vec{v}_{A/O} + \vec{v}_{G/A}\right)}{dt}\right)_{O_{xyz}} = \left(\frac{d\left(\dot{Y}\hat{J} + \omega\hat{k} \times -L\hat{j}\right)}{dt}\right)_{O_{XYZ}} = \left(\frac{d\left(\dot{Y}\hat{J} + \omega L\hat{i}\right)}{dt}\right)_{O_{XYZ}} = \ddot{Y}\hat{J} + \dot{\omega}L\hat{i} + \omega^2 L\hat{j}$$

$$\vec{r}_{G/A} \times M\vec{a}_{G/O} = -L\hat{j} \times \left(\ddot{Y}\hat{J} + \dot{\omega}L\hat{i} + \omega^2 L\hat{j}\right) = L^2\dot{\omega}\hat{k} - L\hat{j} \times \ddot{Y}(\sin\theta\hat{i} + \cos\theta\hat{j}) = (ML^2\ddot{\theta} + M\ddot{Y}\sin\theta)\hat{k}$$

$$\sum_i \vec{\tau}_{i/A} = \frac{d\vec{H}_{/G}}{dt} + \vec{r}_{G/A} \times M\vec{a}_{G/O} = -MgL\sin\theta\hat{k} = (ML^2\ddot{\theta} + M\ddot{Y}\sin\theta)\hat{k}$$

$$\Rightarrow ML^2\ddot{\theta} + ML(\ddot{Y} + g)\sin\theta = 0$$

This is the same result as before, but with far fewer steps and no terms that cancel after all the effort of evaluating them.

Bottom line: Equation (7) is worth remembering as a substitute for Equation (1) which is usually cited in text books. Equation (7) has another directly useful application. It may be applied to problems of the kind: A box sits on a cart that is accelerating will value **a.** Will the box tip over? Equation (7) simplifies the conceptual issues that make such problems often difficult to set up.