Kinematics for 2.003, Fall 2010

1 Some Basics on Frames and Derivatives of Vectors

Kinematics is all about *reference frames*, *vectors*, *differentiation*, *constraints* and *coordinates*.

1. A reference frame is a perspective from which a system is observed. We represent frames with a single letter[1], say O.

2. It is customary to attach three mutually perpendicular unit vectors to each frame. They are assigned names like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ or $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ or $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_t$.[2]

   - We underline vectors on the blackboard, but using a **bold** symbol is the equivalent in type-set form.
   - Consider a situation in which there are frames with unit vectors in each. Vectors can be mixed; an equation like $\mathbf{p} = 3\mathbf{i} + 7\mathbf{e}_\theta$ is perfectly valid.

3. Differentiation of vectors only has meaning if you specify the frame with respect to which you are differentiating them. So it is important to write $\left(\frac{d}{dt}(3\mathbf{a} + 7\mathbf{b})\right)_{/O}$ instead of just $\frac{d}{dt}(3\mathbf{a} + 7\mathbf{b})$ if you are differentiating with respect to a frame O[3]. Often, authors will not identify the frame explicitly, and thus it must be inferred from the context in which the expression is written. For clarity, please always use explicit notation.

   If $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are vectors fixed in frame O, then

$$\left(\frac{d\mathbf{I}}{dt}\right)_{/O} = 0; \quad \left(\frac{d\mathbf{J}}{dt}\right)_{/O} = 0; \quad \left(\frac{d\mathbf{K}}{dt}\right)_{/O} = 0. \tag{1}$$

   A non-zero derivative occurs when the vector is stretching (in which case it is stretching in any frame) or rotating with respect to O. So in general we can say that:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{/O} \neq 0. \tag{2}$$

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[1] Hibbeler refers to frames using three letters corresponding to the three coordinate axes, say XYZ. We will use one letter, often the same as some point fixed in the frame, for notational convenience.

[2] While symbols like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ suffice for simple problems, they may not be sufficient for complex problems involving many reference frames. A systematic method for naming unit vectors associated with a frame is to use the lower case version of a frame's letter along with subscripted numbers. That is, the unit vectors for frame A could be $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. The coordinates associated with these unit vectors can be represented with the same letter and subscripts, but without being designated as a vector (with bold type, a hat, or underline), like a_1, a_2, a_3 instead of x, y, and z. However, we will stick with Hibbeler's notation in this guide.

[3] The subscripted $/O$ can be read as 'with respect to O'. Another popular notation useful when the subscript is needed for other purposes is the pre-superscript, like ${}^O\frac{d\mathbf{a}_1}{dt}$. An even more convenient notation, especially when writing by hand, is $\overset{O}{\mathbf{a}}_1 = 0$. Just as a dot written over a scalar quantity is used to designate time differentiation, so a letter written over a vector quantity can be used to designate time differentiation in a particular frame.

The implication is that if you want to differentiate an expression with vectors in it with respect to a frame of reference O, it is best to express all vector terms in terms of $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ and then use Equation 1 above. That way, taking the derivative becomes trivially easy.

Furthermore, for any two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\left(\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b})\right)_{/O} = \mathbf{a}\cdot\left(\frac{d\mathbf{b}}{dt}\right)_{/O} + \left(\frac{d\mathbf{a}}{dt}\right)_{/O}\cdot\mathbf{b}$$

and

$$\left(\frac{d}{dt}(\mathbf{a}\times\mathbf{b})\right)_{/O} = \mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\right)_{/O} + \left(\frac{d\mathbf{a}}{dt}\right)_{/O}\times\mathbf{b}.$$

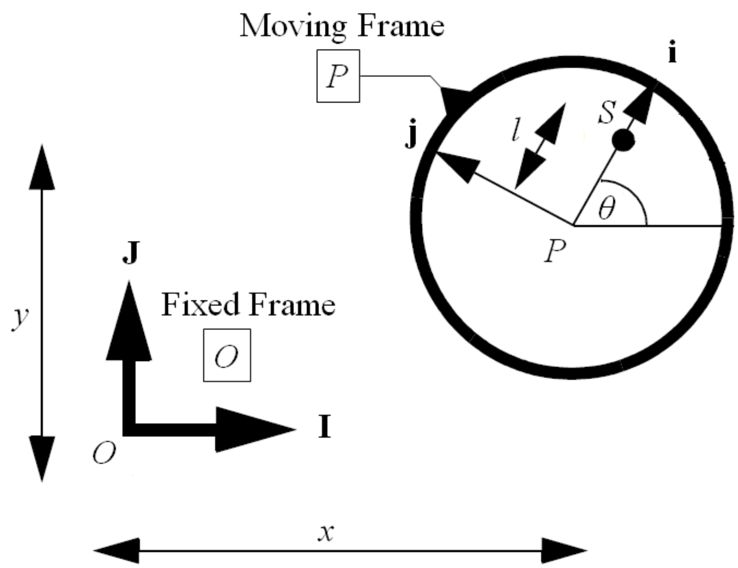

Figure 1: Spider on a Frisbee Problem

2 Calculating Velocities and Accelerations of Given Points

We will now compare three approaches to the classic spider-on-the-frisbee problem (see Figure 1) in which we want to calculate the derivatives of the spider's position with respect to frame O.

2.1 A tedious (but conceptually simple) approach

1. Write the position vector of the spider at point S with respect to point O: $\mathbf{r}_{S/O} = \mathbf{r}_{S/P} + \mathbf{r}_{P/O}$. For convenience, we write it in terms of unit vector components: $\mathbf{r}_{S/O} = x\mathbf{I} + y\mathbf{J} + l\mathbf{i}$.

2. We want to calculate velocities and accelerations of the spider with respect to frame O. Then we want:

$$\mathbf{v}_{S/O} = \left(\frac{d}{dt}\mathbf{r}_{S/O}\right)_{/O} = \left(\frac{d}{dt}(x\mathbf{I} + y\mathbf{J} + l\mathbf{i})\right)_{/O}. \tag{3}$$

3. Because of Equation 1, this simplifies to

$$\mathbf{v}_{S/O} = \mathbf{I}\left(\frac{d}{dt}x\right)_{/O} + \mathbf{J}\left(\frac{d}{dt}y\right)_{/O} + \left(\frac{d}{dt}l\mathbf{i}\right)_{/O}. \tag{4}$$

The first two terms are done. For the third term, we simply rewrite $\mathbf{i}$ in terms of $\mathbf{I}$ and $\mathbf{J}$:

$$\mathbf{i} = \mathbf{I}\cos\theta + \mathbf{J}\sin\theta. \tag{5}$$

We can insert the expressions for $\mathbf{i}$ and $\mathbf{j}$ into Equation 4 above, take the derivative, and we are done:

$$\mathbf{v}_{S/O} = \mathbf{I}\dot{x} + \mathbf{J}\dot{y} - \mathbf{I}l\dot{\theta}\sin\theta + \mathbf{J}l\dot{\theta}\cos\theta + \dot{l}(\mathbf{I}\cos\theta + \mathbf{J}\sin\theta). \tag{6}$$

Note that scalars like x and y have the same time derivative in any frame, so we can indicate derivatives with the usual dot notation.

The acceleration, $\mathbf{a}_{S/O}$, is simply the derivative of $\mathbf{v}_{S/O}$ done exactly as before. Since we have already expanded everything in terms of $\mathbf{I}$ and $\mathbf{J}$, the task of differentiating again is simple. When everything is done, we get

$$\begin{aligned}\mathbf{a}_{S/O} &= \left(\frac{d}{dt}\mathbf{v}_{S/O}\right)_{/O} \\ &= (\ddot{x}\mathbf{I} + \ddot{y}\mathbf{J}) + \ddot{l}(\mathbf{I}\cos\theta + \mathbf{J}\sin\theta) + 2\dot{l}\dot{\theta}(-\sin\theta\mathbf{I} + \cos\theta\mathbf{J}) \\ &\quad + l\ddot{\theta}(-\sin\theta\mathbf{I} + \cos\theta\mathbf{J}) - l\dot{\theta}^2(\cos\theta\mathbf{I} + \sin\theta\mathbf{J}). \end{aligned} \tag{7}$$

Now that differentiation is done, we can rewrite the $\mathbf{I}$ and $\mathbf{J}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. We choose to rewrite some terms in the right-hand-side of Equation 8 above using the following reverse transformations:

$$\mathbf{I} = \mathbf{i}\cos\theta - \mathbf{j}\sin\theta \tag{8}$$

$$\mathbf{J} = \mathbf{i}\sin\theta + \mathbf{j}\cos\theta \tag{9}$$

to get:

$$\mathbf{a}_{S/O} = \underbrace{(\ddot{x}\mathbf{I} + \ddot{y}\mathbf{J})}_{1} + \underbrace{\ddot{l}\mathbf{i}}_{2} + \underbrace{2\dot{l}\dot{\theta}\mathbf{j}}_{3} + \underbrace{l\ddot{\theta}\mathbf{j}}_{4} - \underbrace{l\dot{\theta}^2\mathbf{i}}_{5}. \tag{10}$$

Let's take a closer look at each of the terms in Equation 11:

Term 1 is $\mathbf{a}_{P/O}$: the acceleration of point P with respect to frame O.

Term 2 is $\mathbf{a}_{S/P}$: the acceleration of point S with respect to frame P.

Term 3 is perhaps the least intuitive term, the 'Coriolis' acceleration

Term 4 is the 'Eulerian' acceleration, the angular acceleration scaled by the lever arm l

Term 5 is the familiar 'Centripetal' acceleration

2.2 Taking Derivatives using Angular Velocity and the Transport Theorem

We now introduce the concept of an angular velocity. In 2D, if frame P is rotating with respect to frame O at a rate $\dot{\theta}$ then we say that the angular velocity of P with respect to O is $\boldsymbol{\omega}_{P/O} = \dot{\theta}\mathbf{K} = \dot{\theta}\mathbf{k}$. Some key points about angular velocities:

1. Angular velocities are convenient for two reasons:
   - First, they can be used to calculate derivatives more easily, as we will see shortly.
   - Second, motions of rigid bodies can be captured as the combination of a translation and a rotation. All the points in rigid bodies travel together. So the angular velocity is useful to capture the motion of the entire rigid body. In fact a frame is like a transparent, fictitious rigid body, so angular velocities also capture the rotational motions of frames of reference.
2. A couple of key observations about angular velocities.
   - It might seem that angular velocities need to be defined about a certain point or a axis. This is **<u>not</u>** the case.
   - We will continue our development in 2D but all results will also apply in 3D.

The following equation, known as the 'Transport Theorem', is a formula for the derivative of <u>any</u> vector $\mathbf{r}$ frame of reference O,

$$\left(\frac{d\mathbf{r}}{dt}\right)_{/O} = \left(\frac{d\mathbf{r}}{dt}\right)_{/P} + \boldsymbol{\omega}_{P/O} \times \mathbf{r}. \tag{11}$$

The implication is very simple. As you know from the previous section, you can always take derivatives by expanding everything in terms of the unit vectors attached to the frame of reference with respect to which you seek to take derivatives, say O. However, this can be tedious if some vectors are better specified in some intermediate frame, say P. Equation 11 lets us take derivatives in frame attached to the frisbee, and simply add in a $\boldsymbol{\omega}_{P/O}$ term to take care of the fact that we wanted to compute a derivative in frame O.

Consider the frisbee problem again, and look at Equation 4, reproduced below:

$$\mathbf{v}_{S/O} = \mathbf{I}\left(\frac{d}{dt}x\right)_{/O} + \mathbf{J}\left(\frac{d}{dt}y\right)_{/O} + \left(\frac{d}{dt}l\mathbf{i}\right)_{/O}.$$

Earlier we were compelled to expand the third term in terms of $\mathbf{I}$ and $\mathbf{J}$. That created a lot of math that it would be convenient to avoid. So using Formula 12 we can instead calculate it as:

$$\left(\frac{d}{dt}l\mathbf{i}\right)_{/O} = \left(\frac{d}{dt}l\mathbf{i}\right)_{/P} + \boldsymbol{\omega}_{P/O} \times l\mathbf{i}.$$

Remember that we defined $\boldsymbol{\omega}_{P/O} = \dot{\theta}\mathbf{K} = \dot{\theta}\mathbf{k}$, so the equation above simplifies to:

$$\left(\frac{d}{dt}l\mathbf{i}\right)_{/O} = \dot{l}\mathbf{i} + l\dot{\theta}\mathbf{j}. \tag{12}$$

We can insert that back into Equation 4 and get Equation 7 without all the intermediate complexity, and in a much more convenient form:

$$\mathbf{v}_{S/O} = \dot{x}\mathbf{I} + \dot{y}\mathbf{J} + \dot{l}\mathbf{i} + l\dot{\theta}\mathbf{j}. \tag{13}$$

Now to get $\mathbf{a}_{S/O}$ we can calculate the second derivative; and we use the Transport Theorem again every time we are confronted with a $\mathbf{i}$ or a $\mathbf{j}$ in the term being differentiated with respect to frame of reference O. So in other words, the Transport Theorem is a very simple tool of convenience. We would get the same acceleration as before.

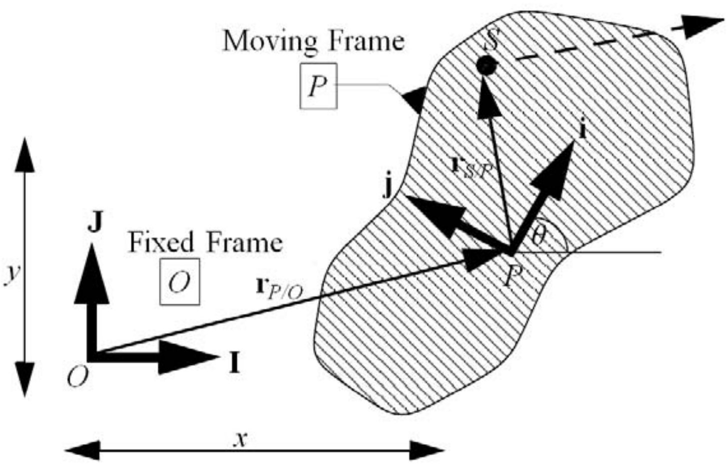

Figure 2: Rigid body hurtling through space

2.3 A Formulaic Approach

We have solved the spider-on-the-frisbee problem using two approaches now. It turns out that the problem has a certain attributes that repeat in a lot of other problems:

1. There are many situations in which a rigid body travels through space either in a ballistic trajectory or under its own power or attached to a mechanism or system by a link or a cable. The rigid body in these situations in moving with respect to a frame O. Let's say that Point O is fixed in Frame O. Let's assume that the rigid body has frame P attached to it.

2. In many cases, there is a point on rigid body, say Point P, whose position is known as a function of time. In other words, $\mathbf{r}_{P/O}(t)$ is known, from which we can calculate $\mathbf{v}_{P/O}$ and $\mathbf{a}_{P/O}$. For example, we might know the exact position of a reference point on a satellite, or the exact location of a navigation unit on a fighter plane, or the exact location of a joint on a link in a robotic arm. This is shown in Figure 2.

3. Assume that the angular velocity of P with respect to O is also known, and we call it $\boldsymbol{\omega}_{P/O}$. We also define the *angular acceleration* of P with respect to O as

$$\boldsymbol{\alpha}_{P/O} = \left(\frac{d}{dt} \boldsymbol{\omega}_{P/O} \right)_{/O}.$$

4. Now consider another point S moving with respect to rigid body P. Say an is astronaut walking on the International Space Station as it hurtles through space. Let us say that you are an observer sitting immobile on rigid body P (the Space Station). In your frame of reference, the point S has a velocity and an acceleration respectively $\mathbf{v}_{S/P}$ and $\mathbf{a}_{S/O}$.

5. Key objective: You want to know what the velocity and acceleration of point S with respect to frame O, $\mathbf{v}_{S/O}$ and $\mathbf{a}_{S/O}$, are. So, to summarize and repeat: we are given $\mathbf{v}_{S/P}$ and $\mathbf{a}_{S/P}$ and we want $\mathbf{v}_{S/O}$ and $\mathbf{a}_{S/O}$.

6. It turns out that $\mathbf{v}_{S/O}$, by a process very similar to our solving the frisbee problem, comes to:

$$\mathbf{v}_{S/O} = \mathbf{v}_{P/O} + \mathbf{v}_{S/P} + \boldsymbol{\omega}_{P/O} \times \mathbf{r}_{S/P}. \tag{14}$$

7. It turns out that the expression for $\mathbf{a}_{S/O}$ can be derived rather trivially, by simply applying the Transport Theorem to Equation 14 to get:

$$\mathbf{a}_{S/O} = \mathbf{a}_{P/O} + \mathbf{a}_{S/P} + \boldsymbol{\alpha}_{P/O} \times \mathbf{r}_{S/P} + \boldsymbol{\omega}_{P/O} \times (\boldsymbol{\omega}_{P/O} \times \mathbf{r}_{S/P}) + 2\boldsymbol{\omega}_{P/O} \times \mathbf{v}_{S/P}. \tag{15}$$

   Judicious use of this formula can save lots of differentiation work.

Using this formula, the frisbee problem can be solved almost trivially. Note that the last three terms are respectively the Eulerian Acceleration, the Centripetal Acceleration and the Coriolis Acceleration as before.

Note though that when using the formula, you must be very careful in specification of frames of reference and of points P and S.

2.4 Summary

There are several key points here.

1. In kinematics, we often calculate velocities and accelerations in moving systems by taking derivatives in frames of reference.

2. There are several ways of calculating these derivatives; three approaches have been discussed.

3. One very important approach is makes use of the Transport Theorem, which relates the derivative in one frame to the derivative in another frame using the concept of angular velocity

4. The angular velocity of a rigid body or frame is a vector quantity with magnitude and direction

5. You can build a very complex system with parts moving on parts moving on parts, and calculate the velocities and accelerations of all the parts by going through this process repetitively. These are called intermediate frames. The angular velocities have a wonderful property:

$$\boldsymbol{\omega}_{P/O} = \boldsymbol{\omega}_{P/Q} + \boldsymbol{\omega}_{Q/R} + \cdots + \boldsymbol{\omega}_{M/N} + \boldsymbol{\omega}_{N/O}. \tag{16}$$

   In other words, angular velocities add over intermediate frames.

6. While this class is limited to 2D, the formulae in this document extend to 3D.