PROFESSOR: Are we ready with the concept questions from the homework this week? How do we get different-- there we go. I looked at one, it was one thing. I looked at the other, [INAUDIBLE].

Does g enter into the expression for the undamped natural frequency? And most people said no, but about a third of you said yes. If you have worked on that problem now, you have already discovered the answer. So you'll find that g does not come into the expression.

When you do a pendulum, g is in the expression. And there's a question on the homework about what's the difference. How can you predict when g is going to be involved in a natural frequency expression and when it is not? I want you to think about that one a bit, maybe talk about it at-- if there's still questions about it, we talk about it in recitation on Thursday, Friday.

OK. So does g enter into the expression here? I'm sure you know this simple pendulum, the natural frequency square root of g over l. For a simple mass-spring dashpot, the natural frequency is k over m whether or not it's affected by gravity. So there's something different about these two.

OK, let's go onto the third one. "In an experiment, this system is given initial velocity observed to decay in amplitude of vibration by 50% in 10 cycles. You can estimate the damping ratio to be approximately," what?

Well, it gives a little rule of thumb I gave you last week. I'll go over it again today. 0.11 divided by the number of cycles did decay 50%. So it took 10 cycles, 0.11 divided by 10, 0.01, 1.1% damping. OK, next.

"At which of the three excitation frequencies will the response magnitude be greatest?" You've done oscillators excited by cosine omega t kind of things before. So at the ratio 1, most people said it's where it would be the largest. Why at 1? Anybody want to give a shout here?

What happens when you drive the system at its natural frequency? It's called resonance, and we're going to talk about that today. So it's when the frequency ratio is one that-- and for the system being lightly damped that you get the largest response.

Finally-- which one are we on here? Oh, this one. Can all the kinetic energy be accounted for by an expression of the form IZ omega squared.

By the way, I brought this system if you haven't. So there's a really simple demo, but it has all sorts of-- so in this case, we're talking about that motion. It certainly has some I omega squared kind of kinetic energy, but does the center of gravity translate as it's oscillating?

What's the potential energy in the system? By the way, any time you get an oscillation, energy flows from potential to kinetic, potential kinetic. That's what oscillation is.

So there has to be an exchange going on between kinetic energy and potential energy. And if there's no losses in the system, the total energy is constant. So the kinetic energy's certainly in the motion, but when it reaches maximum amplitude, what's its velocity when it's right here? Zero.

So all of its energy must be where? In the potential. And where's the potential in this system? Pardon? In the string. That's not stored in the string. There's only two sources of potential energy we talk about in this class. Gravity and--

AUDIENCE: Strings.

PROFESSOR: Strings. Well, these strings don't stretch, so there's no spring kinetic energy. We've got potential energy. There must be gravitational potential energy. How is it coming into this system?

So he says when it turns, the center of gravity has to raise up a little bit, and that's the potential energy in this system. The center of gravity goes up and down a tiny bit. So is there any velocity in the vertical direction? Is there any kinetic energy associated with up and down motion?

Yeah, so that doesn't entirely capture it, 1/2 I omega squared. Is it an important amount of energy? I don't know, but there is some velocity up and down. My guess is that it actually isn't important, that the answer is it does move up and down. It has to, or you would not have any potential energy exchange in the system. OK.

Is that it? OK. Let's keep moving here. Got a lot of fun things to show you today. So last time, we talked about response to initial conditions. I'm going to finish up with that and then go on to talking about excitation of harmonic forces.

So last time, we were considering a system like this-- X is measured from the zero spring force in this case. Give you an equation of motion of that sort. And we've found that you could express x of T as a-- I'll give you the exact expression-- x0 square root of 1 minus zeta squared.

Cosine omega damped times time minus a little phase angle. And there's a second term here, v0 over omega d sine omega dt. And the whole thing times e to the minus zeta omega n t.

So that's our response to an initial deflection x0 or an initial velocity v0. That's the full kind of messy expression. There's another way of writing that, which I'll show you.

Another way of saying is that it's in x0 cosine omega dt plus v0 plus zeta omega n x0, all over omega d sine omega d times t e to the minus zeta omega nt, the same exponent.

This is your decaying exponential that makes it die out. And so I just rearranged some of these things. There's another little phase angle in here now. So you have just a cosine term, this proportional x0, and a sine term, which has both v0 and x0 in it.

The x0 term, if damping is small, this term is pretty small because it's x0 times zeta. When you divide by omega d, which is almost omega n, that goes away. So this term, the scale of it is zeta x0. So if this is 1% or 2%, that's a very small number.

I gave you an approximation, which for almost all practical examples that you might want to do, make it approximately sine here. So this is x0 cosine omega dt plus v0 over omega d sine omega dte to the minus zeta omega nt. And this is the practical one.

For any reasonable system that has relatively low damping, even 10% or 15% damping, you get part of the transient decay comes from x0 cosine, the other part v0 over omega d sine. That's what I can remember in my head when I'm trying to do it.

Now the question, the thing I want to address today is what's this useful for? My approach to teaching you vibration is I want you to go away with a few simple, practical understandings so that you can actually solve some vibration problems, and one of them is just knowing this allows you to do a couple things, and we'll do a couple of examples this morning.

By the way, this form, this is A cosine plus B sine expression. And I label them A and B. A and B, they're both of the form A1 cosine omega t plus B1 sine omega t. And you can always add a sine and a cosine at the same frequency.

If I put just any frequency here, they just have to be the same. You can always take an expression like that and rewrite it as some magnitude cosine omega t minus a phase angle. And the magnitude is just a square root of the sum of the squares-- A1 squared plus B1 squared.

And the phase angle is the tangent inverse of the sine term over the cosine term. So you can always rewrite sine plus cosine as a cosine omega t minus v. We use that a lot, and that'll be used a lot in this course.

And then, of course, if this whole thing is multiplied by an e to the minus 8 omega mt, then so is this. OK. OK, what are these things useful for? And we've derived this all for a mass spring system. Is that equation applicable to a pendulum?

So this expression is applicable to any single degree of freedom system that oscillates. You just have to exchange a couple things. So let's think about a simple pendulum.

So our common massless string and a bob on the end, some length L, equation of motion. And this is point A up here. IZZ with respect to A. Theta double dot plus MgL sine theta equals 0. That's the equation of motion. With no damping, that's the equation of motion in this system. Is it a linear differential equation?

And to do the things that we want to be able to do in this course, like vibration with harmonic inputs and so forth, we want to deal with linear equations. So one of the topics for today is linearization. So this is one of the simplest examples of linearization. We need a linearized equation, and we need to remember in a couple of approximations.

So sine of theta, you can do Taylor series expansion. It's theta minus theta cubed over 3 factorial plus theta to the fifth over 5 factorial plus minus and so forth. And cosine of theta is 1 minus theta squared over 2 factorial plus theta to the fourth over 4 factorial, and so forth.

So what we say, what do we mean when we linearize something? So linearization means that we're essentially assuming the variable that we're working with is small enough that the right hand side, an adequate approximation of this function, is to keep only up to the linear terms on the right hand side. For sine, the term raised to the 1 power, that's the linear term.

This is a cubic term, a fifth order term. We're going to throw those away and say, this is close enough. For cosine, it's 1 minus theta squared. We throw these away. The small angle approximation for cosine as it's equal to 1. That's the simplest example of linearization, of a non-linear term.

So when you linearize this equation of motion, we end up with IZZ with respect to A theta double dot plus MGL theta equals 0, and we get our familiar natural frequency for Bob as square root of g over L. So we need linearization to be able to do pendulum problems. Hmm. OK.

Or maybe let's do an example here. I've got a pendulum that we'll do an experiment with this morning. But 30 degrees is about like that. 17 degrees is about like that. That's quite a bit of angle. Is that small in the sense that I'm linearizing this equation?

So 17 degrees happens to be-- I'll have to use this here. That's actually 17.2 degrees equals 0.3 radians. Sine of 0.3 is 0.2955. And if we fill out and look at these terms, the lead term here is 0.3-- so plug in the 0.3-- minus-- and the second term when you cube 0.3 and divide it by 6, the second term is minus-- what is my number here-- 0.0045.

And if you subtract that from this, you get exactly this. So to four decimal places, you only need two terms in this series to get exactly the right answer. This thing out here, this fifth order term, is really tiny.

But the approximation, if we say, OK, let's skip this, we're saying that 0.3 is approximately 0.2955. Pretty close. So up to 17 degrees, 0.3 radians, that's a great approximation. So it's a little high by about, I think it's about 1 and 1/2% high. So for pretty large angles for pendula, that simple linearization works just fine.

OK, once we get it linearized, that equation of motion is of exactly the same form as the one up there. We don't have any damping in it. We could add some damping. We can put a damping in here with a torsional damper-- ct theta dot.

And now that equation is of exactly the same form as the linear oscillator, linear meaning translational oscillator. Have that inertia term, a damping term, a stiffness term. It's a second order linear differential equation, homogeneous linear differential equation, nothing on the right hand side.

Because they're exactly the same form, then the solution for decay, transient decay from initial conditions, takes on exactly the same form except that it has an initial angle, theta 0, and I use the approximation here. Cosine omega dt plus theta 0 dot, the initial velocity, over omega d sine omega dt all times e to the minus zeta omega nt. So that's the exact same transient decay equation, but now cast in angular terms.

And if you wanted to express it as a cosine omega t minus v, then A would be this squared plus this squared square root and the phi would be a similar calculation as we have up there someone here. Just the B term over the A term, tangent numbers.

All right. What's it good for? So I use this, this equation gets used quite a lot. It has some practical uses. Let's do an example, a little more complicated pendulum.

Draw a stick maybe. Center of mass there. A, IZZ with respect to A. We'll call it ML cubed over 3 for a slender rod.

And now, what I want to do is I have coming along here a mass, a bullet, that has mass m. Has velocity vi for initial here, and that's its linear momentum, p initial. And it's going to hit this stick and bed in it. So you've done this problem before. Yeah.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Pardon?

AUDIENCE: ML cubed [INAUDIBLE].

PROFESSOR: ML cubed over 3. Good. I don't know why I was thinking cubed this morning. ML squared over 3. Good catch there. OK, so we have it. This is mass moment of inertia. This is a pendulum. This bullet's going to come along.

So this is exactly what I've got here. I'll get it in the picture. Yeah. So it's initially at rest. Coming along, this paper, this clip here, it represents the bullet.

So it's swimming along. Hits this thing, sticks to it, and when it hits it, it does that. So then this thing after it hits swings back and forth.

So what's the response of this system to being hit by the bullet? Well, I claim you can do it entirely by evaluating response to initial conditions. But we need to use one conservation law to get there.

So what's conserved on impact? Is linear momentum conserved on impact? How many think yes? Linear momentum conserved. How many think angular momentum's conserved? Hm. Good. you guys have learned something this year. That's great.

Why is linear momentum not conserved? Because are there any possible other external forces on the system? At the pin joint. You can have reaction forces here and there. You have no control of them.

But the moments about this point, are there any external moments about that point during the impact? No. They're reaction forces, but there's no moment arm. So there's no moments.

So you can use conservation of angular momentum. So H1 I'll call it here with respect to A is just R cross Pi. And the R is the length in the I direction.

P is in the j direction, so the momentum is in the k. So this should be mv initial times L, and its direction is in the k hat direction. So that's the initial angular momentum of the system with respect to this. This has no initial angular momentum because it's motionless.

And since angular momentum is conserved, that H2 we'll call it with respect to A has got to be equal to H1 with respect to A, and that will then be IZZ with respect to A theta dot. But I need to account for the mass, this thing. So the total mass moment of inertia with respect to A is IZZ with respect to A plus M-- what?

Now I've got the total mass moment of inertia with respect to this point, that of the stick plus that of the initial mass that I've stuck on there. And this must be equal to theta dot. And I put a not down here because I'm looking for my equivalent initial condition. And this then is mv initial times L. Then I can solve for theta dot 0, and that looks like mv initial L over IZZ A plus mL squared. And everything on the right hand side you know.

You know the initial velocity, the mass of the bullet, the length of the distance from the pivot, mass moment of inertia, and the additional mass moment of inertia. These are all numbers you plug in, and you get a value for this. And once you have a value for this, you can use that.

In this problem, what's theta 0? The initial angular deflection at time t0 plus right after the bullets hit it. And it hasn't moved because it hasn't had time to move yet.

At some velocity, it takes finite time to get a deflection. So there's zero initial angular deflection, but you get a step up in initial angular velocity. And so the response of this system is theta t is theta 0 dot over omega d sine omega dt.

So what's omega d? Remember, I'll define a few things. In this case, this is ct over 2 IZZ z A plus little ml squared-- we have to deal with all the quantities after the collision-- 2 times omega n. That's the damping ratio for this torsional pendulum, with this pendulum. It's the damping constant. 2 times the mass, the inertial quantity, times omega n for a translational system at c over 2 m omega n.

For a pendulum system, it's the torsional damping over 2 times the mass moment of inertia times omega n. And omega n, well, it is going to calculate the natural frequency. It's just MgL divided by IZZ plus this. Maybe you ought to write that down.

So always for a sample singular [INAUDIBLE] oscillator, you want the undamped natural frequency. Ignore the damping term. Take the stiffness term coefficient here and divide it by the inertial coefficient. But we care about the natural frequency after the impact, so this is going to be-- ah. The trouble is here I don't know for this system, I haven't worked out yet, what this term looks like.

What is it? This result right here is for the simple Bob. For this stick, it's MgL over 2 plus the little m times l. Little more messy.

So MgL over 2 plus little ml. That'll be the-- come from the potential energy in this system all over IZZ A plus mL squared. So you get your natural frequency out of that expression. OK.

So you do this problem sometimes before when you do, say, somebody asks you how high does it swing. AND so forth. Well, you can do it by conservation of energy, et cetera. But now, you have actually exact expression for the time history of the thing after the impact, including the effects of damping.

And if you were to draw the result of this function of theta as a function of time, this one starts with no initial displacement but a velocity and does this. And that's your exponential decay envelope, and this is time. Now, what-- yeah. Excuse me.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Why this one?

AUDIENCE: Yes. [INAUDIBLE].

PROFESSOR: Excuse me. Forgot the g. I mean, it accounts for the restoring moment, the additional little bit of restoring moment that you get from having added the mass of this thing to it, right.

So it has by itself MgL sine theta, and we linearize that too. So it's MgL theta, and those two terms would add together. So you just have a second term here that has MgL like behavior.

How I most often personally make use of expressions like this, or the one for translation, is because experimentally, if I'm trying to predict the vibration behavior of a system, one of the things you want to know is the damping. And one of the simplest ways to measure damping is to give a system an initial deflection or initial velocity and measure its decay, and from its decay, calculate the damping.

So the last time I gave you an expression for doing that, and that was a damping ratio 1 over 2 pi times the number of cycles that you count, that you watch it, times the natural log of x of t over x of t plus n periods of vibration. This has a name. It's called the logarithmic decrement, this thing. So if somebody says log decrement, that's where they're referring to this expression.

A comment about this. In this expression, x of t must be zero means if your measurement-- we have x of t here, or theta of t-- they must be zero mean. There must be oscillations around zero or you have to have subtracted the mean to get it there because if this is displaced and is oscillating around some offset, then this calculates and will get really messed up.

It's got an offset plus an offset here plus an offset there. It means it's totally meaningless. So you must remove the mean value from any time history that you go to do this.

So there's an easier way the same expression-- and this is, in fact, the way I use this. A plot out like just your data acquisition grabs it, plots it for you. I take this value from here to here, and this is my peak to peak amplitude. And then I go out n cycles later and find the peak to peak amplitude.

And so this is perfectly, this is just the same as 1 over 2 pi n, but now you do natural log of x peak to peak t over x peak to peak at t plus n periods. And that now, peak to peak measurement, you totally ignore the mean. Doesn't matter where you are. You want the here to here, here to here, plug it in there, and you're done.

OK, so let's-- I got 1, 2, 3, 4. Let's let n equal 4, and let's assume this expression here-- n is 1 over 2 pi times 4. And let's assume that in these four periods from-- that's 1 period, 2, 3, 4 getting out here to this fourth, four periods away, that this is one fifth the initial. So this would be the natural log of 5.

So 1 over 2 pi times 4, natural log of 5, and you run the numbers, you get 0.064, or what we call 6.4% damping. That's how you do it. That's the way you do a calculation like that. Now, I gave you a quick rule of thumb for estimating damping, and this is what I-- I can't work.

I don't do logs in my head, but I can do damping estimates without that because I know that zeta is also-- if I just plug in some numbers here and run them all in advance is 0.11 divided by the number of cycles to decay 50%. So we're going to do an experiment.

And I guess it can be seen with the camera. So here's my pendulum. This is my initial amplitude, and this is about half. So if I take this thing over here, like that, then let go, and count the cycles that it takes to decay halfway, we can do this experiment.

So let's do it carefully. So line it up like that, and you're going to help me tell-- you count how many cycles it takes till it gets to here. So 1, 2, 3, 4, 5, 6, 7, 8. About eight cycles.

So it decayed halfway in eight cycles. So zeta is approximately 0.11/8. It's 1 and 1/2%, 1.4%, something like that. Perfectly good estimate of damping.

Now, the stopwatch here, we can do this experiment again. I want you to count. You're doing the counting. And I'm going to say start, and I want you to count cycles until I say stop. Now, I'll probably stop on 10 to make the calculation easy, so quietly to yourself count the number of cycles from the time I release it until the time I stop. Come back here.

So this time, the backdrop doesn't matter. I just want you to count cycles. And I'll start-- I'll let it get going, and when it comes back to me is when I'm going to start the stopwatch because I have a hard time doing both at the same time. So start. How many cycles?

AUDIENCE: [INAUDIBLE].

PROFESSOR: So I got 17.84 for 10. So 10 divided by 10 is 1.784 seconds per cycle. Can't write like that. 1.784 seconds per cycle. The frequency would be 1 over that, right.

The thing you have to be careful about when you're counting cycles is if I start here, that's 0, 1, 2. A very common human mistake is when you're counting something like this is to say one when you start, and then you're going to be off by one count.

Follow me? If I start 0, 1, 2. So I start the clock on zero, but the first cycle isn't completed till one whole cycle later. So be careful how you count. OK.

Now we're going to shift gears and take on a new topic, and that's the response to a harmonic input, some cosine omega t excitation. Yeah.

AUDIENCE: [INAUDIBLE].

PROFESSOR: What is omega d? So it's the damped natural frequency. That's how it's referred to. It is the frequency you observe when you do an experiment like we just did. The actual oscillation frequency when it's responding to initial conditions is slightly different from the undamped, but if you have light damping, if you have even 10% damping, 0.1 squared is 0.01.

That's 0.99 square root. It's 0.995. So you're only off by half. They're only half a percent difference. So for lightly damped systems, for all intents and purposes, mega n and mega d are almost exactly the same.

OK. We now want to think about-- we have a linear system putting a force into it. It looks like some F0 cosine omega t. And out of that system, we measure a response, x of t. And inside this box here is my system transfer function. It's the mathematics that tells me I can take F of t in and predict what x of t out is.

So I need to know the information that goes into this box, and of course, the real system-- this is just the mechanical system. Force in, measured output out. This is what we call a single input single output system, SISO, single input single output linear system. And there's all sorts of linear systems that you're going to study as mechanical engineers, and you've already begun, I'm sure, studying some of them.

One of the properties of a linear system is that you put a force in, F1, and measure a response out, x1. And then you try a different force, F2, and you measure a response out, x2. What's the response if you put them both in at the same time, F1 and F2? You just add the responses to them individually.

So F1 gives you x1. F2 give you x2. F1 plus F2 gives you x1 plus x2, and that's one of the characteristics of a linear system. We use that concept to be able to separate the response. Our calculation's about the response of a system, like our oscillator here, separate its response to transient effects, transience being initial conditions. They die out over time-- that's why we call them transients-- and steady state effects.

So cosine omega t, you can leave it running for a long, long time, and pretty soon, the system will settle down to responding just to that cosine omega t. And that we call steady state. And we use them separately. So we've done initial conditions. Now we're going to look at the steady state response of a-- say our oscillator, our mass spring dashpot, to a harmonic input, F0 cosine omega t.

Another brief word. If I have a force, F0 cosine omega t would look like that. And the response that I measure to start off with my-- it's sitting here at zero when you turn this on. And it's going to do some odd things initially, and then eventually settle down to some long term steady response.

The amplitude stays constant. It stays angle with respect to the input isn't necessarily the same. There's some possibly phase shift. And that's so the two, if you're plotting them together, they won't line up.

But see this messy stuff at the beginning? When you first turn this on, it jumps from here to here, that force, and it gives it a kick to begin with. And this will have some response initially due to that transient start up. And this response is all modeled by the response to initial conditions.

And it'll die out after a while, this messy stuff. What's the frequency? What frequency do you expect this initial, erratic looking stuff to be at? Its response to initial conditions.

What is the model for a response to initial conditions? What's the frequency of the response to initial conditions of the single degree of freedom system? We have an equation over here, right?

The top has a cosine term and a sine term. Part of it's a response to initial displacement. Part of it's a response to the initial velocity. Any of this start up stuff can be cast as initial conditions, and the response to initial conditions is always at the natural frequency period. No other frequencies for same degree of freedom systems.

So you get a behavior that's oscillating at its natural frequency. Mixed in there is a response at the excitation frequency. And after a long time, the response is only excitation frequency. This is now out here. This is omega.

In here, you have omega and omega d going on. So this is messy. Usually isn't important, but it is. There are ways of getting the exact solution, but mostly, vibration engineers, you're interested in the long term steady state response to what we call a harmonic input. OK.

So we'll work a classic single degree of freedom oscillator problem-- excited by F0 cosine omega t. You've done this in 1803, but now we'll do it using engineering terminology. We'll look at it the way a person studying vibration would think about this. We know the equation of motion.

And I'm interested in the steady state response. So this is x, and I'll do-- you just write it once like this. SS, steady state. I'm only interested in its-- after those transients have died out. And that steady state response I know is going to be some amplitude X0 cosine omega t minus some phase angle that I don't necessarily know to begin with.

But that's my input. This is my output. I plug it into here and turn the crank and see what falls out. So you plug both of those in, and you get two-- you get-- this is going to be a little writing intensive for a few minutes.

So you plug the X0 cosine omega t into all of these terms. The m term gives you minus m omega squared, the k term gives you a k, and the damping term, minus c omega sine omega t minus v. All of that equals the right hand side-- F0 cosine omega t. So this just purely from substitution and then gathering some terms together.

I'm going to divide through by k, by k. If I divide through by k, k divided by-- this gives me a one. This gives me an m over k, which is 1 over the natural frequency squared, for example. And I'm going to put this into a form that is the standard form for discussing vibration problems.

So this equation can be rewritten in this form. 1 minus omega squared over omega n squared cosine omega t minus v minus 2 zeta omega over omega n sine omega t minus v. All that's still equal to F0 cosine omega t.

So this is getting into kind of more standard form. So there's 1 minus omega. This now, this omega over omega n, is called the frequency ratio, and you see a lot of that.

And I've substituted n here. c omega over k turns out to be 2 zeta omega over omega n. So this frequency ratio appears a lot. in our-- let's see here. You need a couple of trig identities-- cosine omega t minus v. Cosine omega t cosine phi plus sine omega t sine phi, and sine omega t minus phi gives you [INAUDIBLE] sine.

Sine omega t cosine phi minus cosine omega t sine phi. So that's a trig identity you actually use quite a bit doing vibration problems. We need them, so we take these, plug them in in all these places, and do quite a bit of cranking. Yep.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah. Thank you. And that's called, that f over k, is how much the spring would move statically, at which the point would move statically. We'll need that term also. OK. You do all of this. Here, I'll call these--

OK. So this is C. That's expression C.

I can't see that probably. Call this D, this E. So you plug D and E into C and work it through, you get two equations. You break it into two parts because one is a function of cosine omega t, and then you have another part after this substitution that's a function of sine omega t, and you can separate them.

But there's no sine omega t force. On the right hand side, you get zero. There are two equations here. How many unknowns do we have? All we know when we start this thing is the input, and we have unknown response amplitude, and we have an unknown phase that we're looking for.

How many equations? How many unknowns? Two and two. So you can do a lot of cranking, which I have no intention of doing here, and solve for the amplitude of the response and the phase.

And every textbook-- the Williams textbook does this. There are two readings posted on Stellar by [? Row. ?] Every textbook goes through these derivations that I've just done. Nick, you've got a question.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Pardon?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, I keep forgetting it. You're right. So we got a k here. And notice, this equation, we throw away that for now. We get rid of this for now. We have these two equations and two unknowns are just algebraic equations There's not time dependent. We can get rid of that part.

So we've now reduced this to algebra, and the answer is plotted up there. You've probably seen it before. It says that x0 is F0 over k-- I can get it right this time from the get go-- over a denominator, which appears again and again and again in vibration. Omega squared over omega n squared squared plus 2 zeta omega over omega n squared square root, the whole thing, and an expression for phi.

Tangent inverse of 2 zeta omega over omega n, 1 minus omega squared over omega n squared. So you can solve all that-- this mess over here for these two quantities. Do you need to remember this?

You ever going to be asked this on a quiz? Not by me. You ever going to have to use this on a quiz and in homework? Absolutely.

So the takeaway is today know how to use those response to initial condition formulas and damping and these two. So when you plot, when you plot these, you get this picture up there. And we need to talk about the properties of this.

Remember, this omega over omega n is the same called the frequency ratio. It's just the ratio of the excitation frequency to the natural frequency of the system. And when they're equal, for example, this ratio is one. This whole thing in parentheses goes to zero.

This expression over here goes to 2 zeta, because that's one. 2 zeta squared square root is just 2 zeta. When omega equals omega n, this whole expression is F0 over k divided by 2 zeta, for example. And that's called resonance, and that's when you're right at where that peak goes to its maximum.

Let's talk about this expression for a moment. If we have our cart, our mass-spring dashpot we started here. If you apply a force, a static force F0 and stretch the spring by an amount F0 over k. So x-- what we'll call x static is just F0 over k.

And if I want to plot, I want to-- this has a name. This is called, this ratio here, this gives you the magnitude of the response. It goes by a variety of names. Some people call it a transfer function. Some people call it a frequency response function. I write it intentionally this way.

This is I put output over input because this expression has units of output over input. So I just write it like this, remind myself what this transfer function is about. The input is force, the output is displacement. This expression has units of force, force per unit displacement.

If I go to here, if I try to plot this-- let me start over. If I try to plot this, it's going to be depending on the exact value of the spring constant and the exact value of the force every time. I have to get a unique plot every time I go to do this. So textbooks and engineers, I don't want to have to remember this part. This is where all of the content is in is in this denominator, and it's dimensionless.

So what I'd really like to plot is x0 over x static. And if I do that, that is x0 over the quantity F0 over k. If I just divide-- this is x static-- it would bring this to this side, then this expression, this is just 1 over that denominator. And sometimes, I think in the handout by [? Row ?], they just call this h of omega.

It's dimensionless, frequency over frequency, and that's actually what's plotted up there. And this is called-- has different names also. Magnification factor, dynamic amplification factor, because the ratio of x to x static if this is the dynamic effects magnify the response compared to the static response. So it might be this over this might be 10. I mean, the dynamic response is 10 times the static response.

OK, how do you-- to sum this up-- and we'll be kind of getting close to the end. We want to talk just about the properties of this. How do we use this?

So in practical use, you have an input specified, some force cosine omega t. You know you have a single degree of freedom oscillator that is governed by equations like that one, and you want to predict the response. Well, you say x of t is equal to the magnitude of the force times the-- and you divide that by-- we could do it this way.

The magnitude of the force divided by k, which is the static response. To predict x0, we just have to predict this quantity, multiply it by F0 over k. So you know this. You better know that about your system, and you multiply it by this quantity magnitude of h of omega.

And the time dependent part is times cosine omega t minus the phase angle. And the phase angle, you get either off the plot or from the-- have I written it down? I haven't written the phase angle down yet.

It's kind of a messy expression too. That's why we plot it. 2 zeta omega over omega n over 1 minus omega squared over omega n squared.

But by knowing just this plot, what you just put in every textbook about vibration in the world, by knowing this magnification factor, calculating the static response, multiplying the two together, you have the amplitude of the response, and its time dependence is cosine omega t minus the phase angle. And I've got a little example here. Actually, rather than the example, I've gone to all the trouble of setting this up.

All right. This is just a beam. Where's my other little beam? And a beam is just a spring. Put a mass on the and.

This is basically a single degree of freedom system. It has a natural frequency. The beam has a certain stiffness. And now, in this case, we're interested in response to some harmonic input. So any of you know what a squiggle pen is. This is a kid's toy.

AUDIENCE: Excuse me. Can you move the camera a tad to the left so the [INAUDIBLE]?

PROFESSOR: Yeah. So all throughout the term, we've studied rotating masses quite a bit, right. This thing has a rotating mass that you can see in the end. I mean, when you leave it, you can come down and take a look. It has a low rotating mass. It's actually a pen, but it's a kid's toy. Shakes like crazy.

And now we need the lights down. And it happens that the-- I've got a strobe light here, and I've kind of preset the frequency so it's very close to the frequency of vibration of this beam. So there's a rotating mass in this pen going round and round, and it puts a force into the system that looks like F0 cosine omega t in the vertical direction.

Also does it in the horizontal, but vertical is our response direction. So it's putting in a force, and I have set the length of this beam so that the natural frequency of this beam with this mass on the end is exactly very close to being the frequency of the excitation. And the flash rate is slightly different than the vibration rate, so you see it illuminated at many positions as it goes through the cycle. So you see it going up and down.

So if I mismatch it quite a bit, then you see it going. And actually, if you look at the very right end, you can see a white thing going up and down. That's the mass where you can actually see the mass in the very end of the-- right there.

You can see something going around and round. There, that's the rotating mass. So the beam is going up and down, and I've got this, the vibration frequency of that mass going round and round equal to the natural frequency of the beam of the mass in the end.

And it's moving quite a bit, and I'll loosen my clamp, and I'm going to change the length of the beam. I've shortened it. And now it's still moving up and down but not as much because the frequency of the rotation of the eccentric mass is no longer close to the natural frequency of the system. In fact, I've made the natural frequency of the system-- you can bring the lights back up-- I've made the natural frequency of the system. By making the beam shorter, I've made it stiffer.

So the natural frequency has gone up. The frequency of the rotation of the eccentric mass has stayed about the same. So what's happened to that frequency ratio, omega over omega n? So less than one or greater than one. So the omega n has gone up. Omega stayed the same.

The frequency ratio when you shorten this beam is less than one, and the properties of this transfer function, we call it-- this magnification factor looks like this. When we're exciting it right at one-- this is omega over omega n-- you write it resonance. When you excite it at a frequency ratio less than one, you start dropping off this backside, and the response goes down. And if you excite it at frequencies much greater than the natural frequency, you end up way out here. I can do that too.

So how much you think it will vibrate now? A lot? A little? Hardly-- oops. Oh, I've brought it out so much you can't see it.

Hardly moving at all, and that's because in terms of this terminology of magnification factors, transfer functions, this is a plot of x over x static. It goes right here. When you go to zero frequency, you are at static, so the response at very, very low frequency goes to being the same as the static frequency. So in this plot, it goes to one.

At resonance, you put in one here. This goes to zero. That becomes a one 2 zeta squared. This height here is 1 over 2 zeta. So the dynamic amplification at resonance is just 1 over 2 times the damping ratio.

You have 1% damping. Twice that is o2. 1 over o2 is 50.

So if you only had 1% damping, the dynamic amplification, the amount that this vibrates greater than its static response, is a factor of 50 greater. But then as you go higher in this omega over omega n, and you get way out here, and you get almost no vibration at all. And that's what's happened when I've lengthened this.

OK. So there's your introduction to linear systems. In this case, a single degree of freedom system that vibrates an oscillator, we're talking about steady state response, not the part of the solution, the mathematical solution, to the initial conditions. So all of this has been about steady state response of our simple oscillator to what we call a harmonic input, F0 cosine omega t.

So what's important that you need to remember and be able to use? This concept here, this idea of a transfer function. That's really important.

You might want to remember it though as this dimensionless quantity x over x static. Just remember the shape of this transfer function. Magnitude of the amplification, 1 over 2 zeta at resonance. And it goes to one at low frequency. The high frequency, it drops away off.

Next time, we're going to pick up the topic of what we call vibration isolation. The practical thing to know as engineers is when you have a significant vibration problem, like in a lab, and you're looking through your microscope, and the floor vibration is causing trouble with your microscope, and you can't move the subway, what can you do to solve that problem?

Well, you might be able-- what if you put some kind of a flexible pad under the microscope? You might be able to reduce the vibration of the microscope. Things like that. That is the topic of vibration isolation, so we're going to get into that next time. Thanks.