2.003 Engineering Dynamics
Problem Set 9--Solution

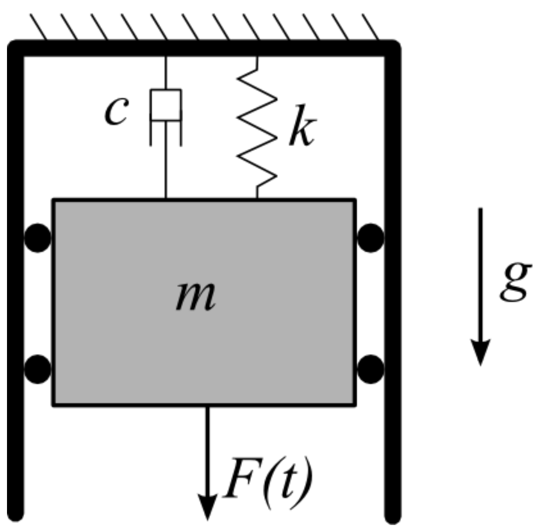
Problem 1

Find the equation of motion for the system shown with respect to:

a) Zero spring force position. Draw the appropriate free body diagram.

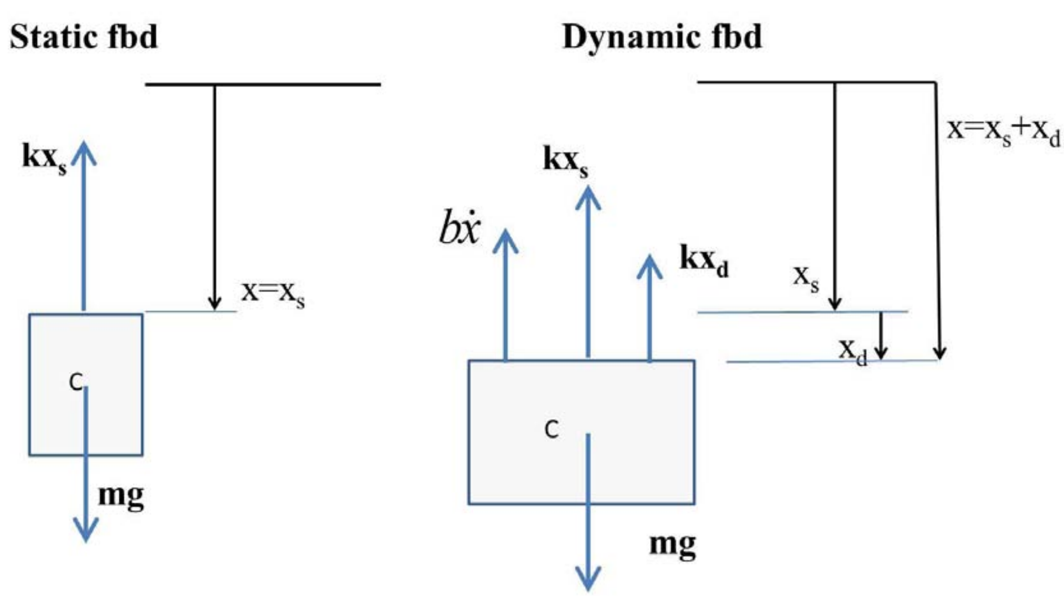
b) Static equilibrium position. Draw the appropriate free body diagram.

Solution: Let x and x_d be the displacements of the mass, measured from the zero spring force and the static equilibrium positions, respectively. Let x_s be the static displacement under the influence of gravity. These metrics are related as follows:



$$x = x_s + x_d$$
$$\dot{x} = \dot{x}_d \qquad \textbf{(1)}$$
$$\ddot{x} = \ddot{x}_d$$

The free body diagrams are shown below. The coordinate x is measured from the zero spring force position. When hanging in the static equilibrium position $x = x_s$ and $x_d = \dot{x}_d = \ddot{x}_d = o$. The static spring force must be equal and opposite to the weight. Therefore, $mg\hat{i} - kx_s\hat{i} = o$, and $mg=kx_s$. (2)



For the case with additional dynamic motion, from the free body diagram sum the external forces and equate to the mass times the acceleration:

$m\ddot{x} = -kx - b\dot{x} + mg$, which upon rearrangement becomes
$m\ddot{x} + b\dot{x} + kx = mg$, which is the EOM based on the displacement measured from the zero spring force position. (3)

If x and its derivatives in (3) are replaced by the equivalent expressions from (1) in terms of the static and dynamic components of the motion the following EOM is obtained.

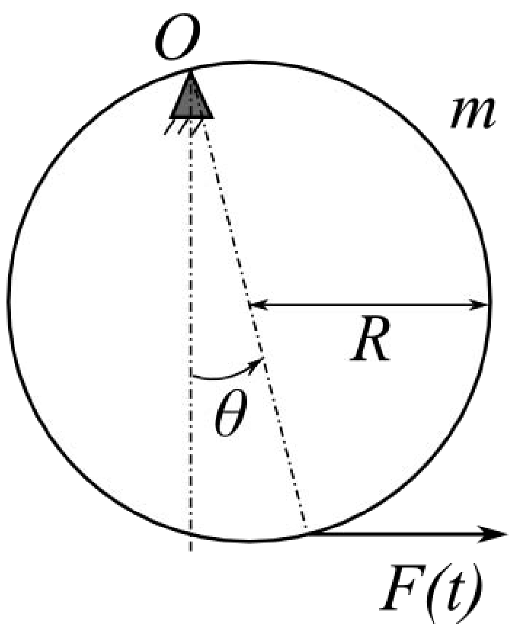
$m\ddot{x}_d + b\dot{x}_d + k(x_s + x_d) = mg$, which upon rearrangement becomes
$m\ddot{x}_d + b\dot{x}_d + kx_d = mg - kx_s = 0$ from (2)
$m\ddot{x}_d + b\dot{x}_d + kx_d = 0$, the EOM expressed with respect to the static equilibrium position.

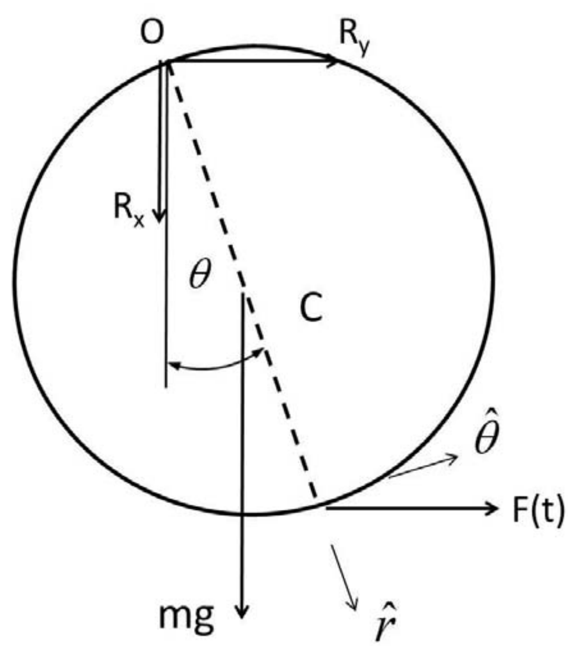
The key point to understand is that the spring force due to the static deflection exactly cancels the weight of the mass, regardless of the amount of the dynamic motion, which is in addition to the static deflection. In this case the term involving gravity is a constant, mg. It does not vary with dynamic motion. Whenever this happens in a dynamics problem, by writing the EOM with respect to the static equilibrium position, the term involving gravity can be eliminated from the EOM. This is the answer to problem 3 as well.

Problem 2

A thin hoop of mass m and radius R is hanging from a knife edge (there is no slip between the hoop and the knife edge). The excitation force, F(t), is always horizontal.

a) Draw a free body diagram and derive the equation of motion of the hoop.





b) Find the linearized equation of motion using a small angle approximation for θ (define θ such that the static equilibrium position is θ=0).

c) Find the undamped natural frequency of the system.

Solution: This is a planar motion problem. See the free body diagram above. Summing the torques with respect to the fixed point 'O' allows us to write Euler's law in the simplified form:

$$\sum_{external} \vec{\tau}_O = I_{zz/O}\ddot{\theta}\hat{k} = R\hat{r} \times -mg\sin\theta\hat{\theta} = -mgR\sin\theta\hat{k}$$

$\Rightarrow I_{zz/O}\ddot{\theta} + mgR\sin\theta = 0,$ where $I_{zz/O} = I_{zz/G} + mR^2,$ using the parallel axis theorem.

For a thin hoop, $I_{zz/G} = mR^2$ and therefore

$I_{zz/O} = mR^2 + mR^2 = 2mR^2,$ which upon substitution yields

$2mR^2\ddot{\theta} + mgR\sin\theta = 0.$

The small angle approximation $\sin\theta \bullet \theta$, allows the equation of motion to be linearized:

$m\ddot{\theta} + \frac{mg}{2R}\theta = 0.$ Assuming a solution of the form

$\theta(t) = A\cos(\omega t)$ results in an expression for the natural frequency:

$\omega_n = \sqrt{g/2R.}$

Problem 3

Come up with a general rule that will predict when the acceleration of gravity will appear in the expression for the natural frequency. Compare the systems of problems 1 and 2 to illustrate your answer.

Solution: In problem 1 the term in the EOM involving gravity was mg. In equation 2 the term in the EOM involving g was $mgR\sin\theta$. In problem 2, the term involving gravity also involved the motion coordinate, θ. Whenever the gravity term in the equation of motion is not a function of the motion coordinate, then by using a coordinate which is measured from the static equilibrium position will eliminate the gravity term from the EOM. In such cases the natural frequency is not a function of gravity. When the gravity term involves the motion coordinate, then the natural frequency will also involve gravity, as occurs in problem 2.

Problem 4

For the system in problem 1, let K=10,000 N/m, M=0.633 kg, x_o=0.1m, v_o=10 m/s and the damping ratio ζ =0.05.

a. Find an expression for x(t) in terms of the initial conditions. Express x(t) in the form $x(t) = A\cos(\omega t - \phi)$.

b. Sketch x(t) versus time.
c. Compute the ratio of the damped to the undamped natural frequency for damping ratios of (i) 5%, (ii) 10% and (iii) 20% of the critical damping.

Solution: The general expression for the response of a single DOF system to initial conditions is:

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) = \left| x_o \sin(\omega_d t) + \frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\cos(\omega_d t) \right|$$

a. where

$$A = \left[x_o^{\,2} + \left(\frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d} \right)^2 \right]^{1/2}, \quad \phi = \tan^{-1}\left[\frac{\dot{x}_o + \zeta\omega_n x_o}{x_o\omega_d} \right] \text{ and } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

For this problem, $x_o = 0.1m$, $v_o = \dot{x}_o = 10m/s$, and $\zeta = 0.05$.

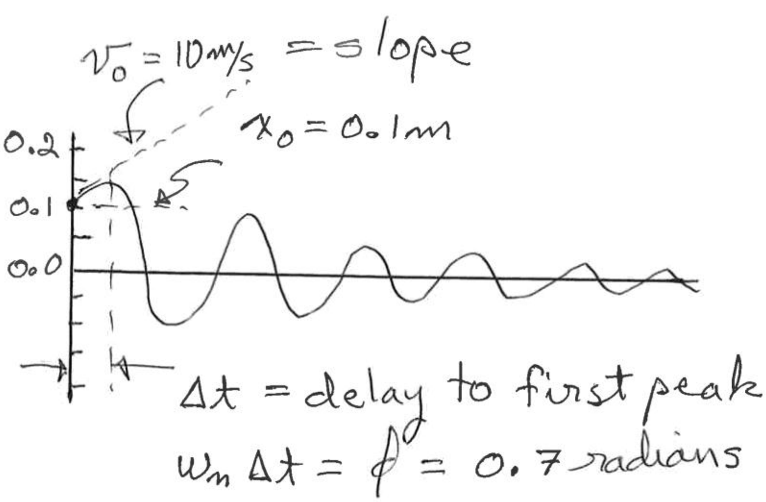
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000N/m}{0.633kg}} = 125.7rad/s,\ \omega_d = \omega_n\sqrt{1-0.05^2} = 0.999\omega_n = 125.6r/s$$

$$\zeta\omega_n = 6.28r/s$$

$$A = \left[0.1^2 + \frac{(10 + 6.28 \bullet 0.1)^2}{125.6^2} \right]^{1/2} = 0.131m$$

$$\phi = \tan^{-1}\left(\frac{(10 + .628)}{0.1 \bullet 125.6} \right) = 0.7radians$$

b.



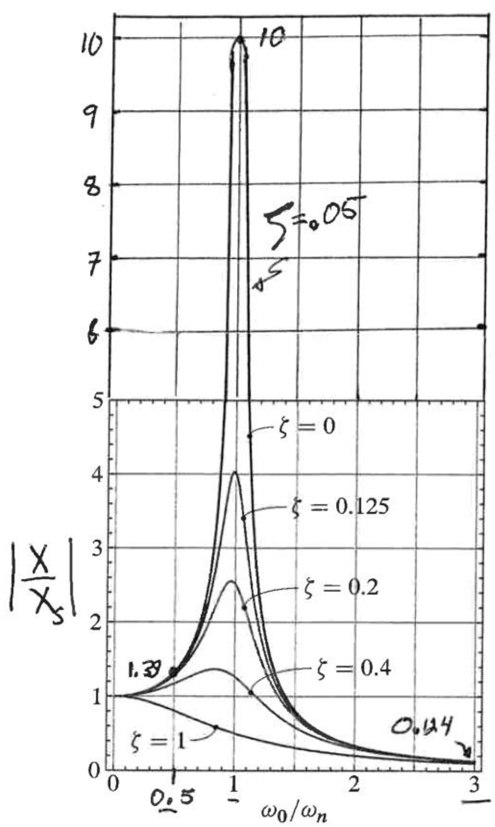
c. $\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2} =$ 0.999 , 0.995, and 0.98 for ζ=0.05, 0.1 and 0.2 respectively.

Problem 5

For the values of M, K used in the previous problem, compute the steady state amplitude and phase angle of the response to a harmonic force specified as $F(t) = F_o \cos(\omega t)$, where F_o=10N and the damping ratio is 5% of critical. Do this computation for three values of ω / ω_n = (i) 0.5, (ii) 1.0 and (iii) 3.0.

Solution: To compute the steady state response of a single DOF system apply the transfer function between the input force and the response displacement.

$$H_{x/F}(\omega) = \left|\frac{x}{F}\right| = \frac{1}{k}\frac{1}{\left[\left(1-\frac{\omega}{\omega_n}^{2}\right)^{2}+\left(2\zeta\frac{\omega}{\omega_n}\right)^{2}\right]^{1/2}}$$



or in terms of the dynamic amplification

$$H_{x/x_S}(\omega) = \left|\frac{x}{x_s}\right| = \frac{1}{\left[\left(1-\frac{\omega}{\omega_n^2}^{2}\right)^{2}+\left(2\zeta\frac{\omega}{\omega_n}\right)^{2}\right]^{1/2}},$$

where $x_s = \frac{F_o}{k} = \frac{10N}{10,000N/m} = 0.001m$

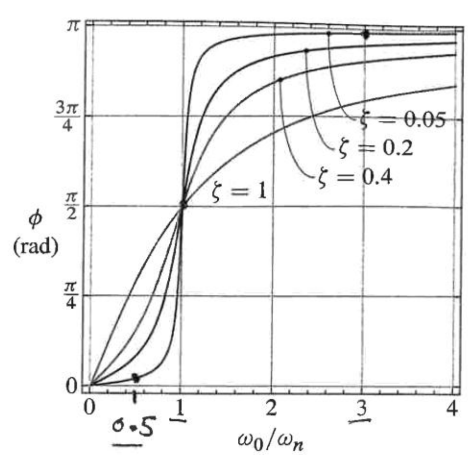
When evaluated at $\frac{\omega}{\omega_n} = 0.5, 1.0 \text{ and } 3.0$

$H_{x/x_S}(\omega) = 1.33,\ 10,\ 0.124,$

$x = x_s H_{x/x_S}(\omega) = 0.00133,\ 0.01,\ 0.000124$

For the 3 cases in order ϕ is given by

$$\phi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega}{\omega_n^2}^{2}} = 0.067,\ \frac{\pi}{2},\ \text{and } 3.1 \text{ radians.}$$



$\left|\frac{x}{x_s}\right|$ and ϕ are shown in the figures to the right.

The response x(t) may be written as

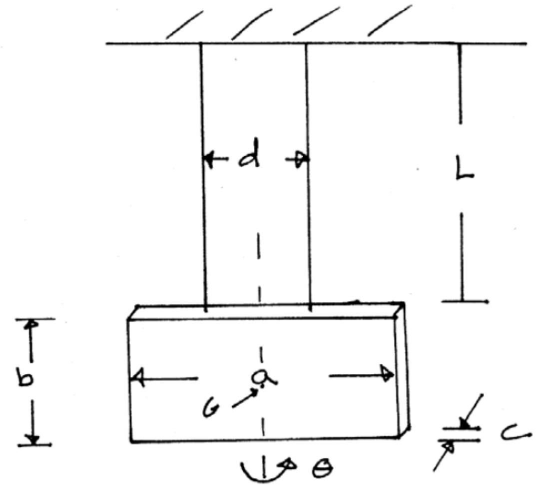
$$x(t) = |F_o|\,|H_{x/F}(\omega)|\cos(\omega t - \phi)$$

$$x(t) = \left|\frac{F_o}{k}\right|\,|H_{x/x_S}(\omega)|\cos(\omega t - \phi)$$

Problem 6

A block of wood is suspended by two strings, as shown in the figure below. The strings are separated by d=10.0 cm. and are 28 cm in length. a= 14.0 cm, b=6.0 cm, and c=1.8 cm.

a) Determine the mass moment of inertia with respect to the vertical axis of rotation which passes through the center of mass of the block.
b) Find an equation of motion for small torsional oscillations about the vertical axis which passes through the center of mass. Use the direct method in which you sum the external moments to find the equation of motion. Linearize the equation of motion for small oscillations.
c) Find expressions for T and V, the kinetic and potential energies of the block in terms of the rotational velocity and the angle of rotation of the block.

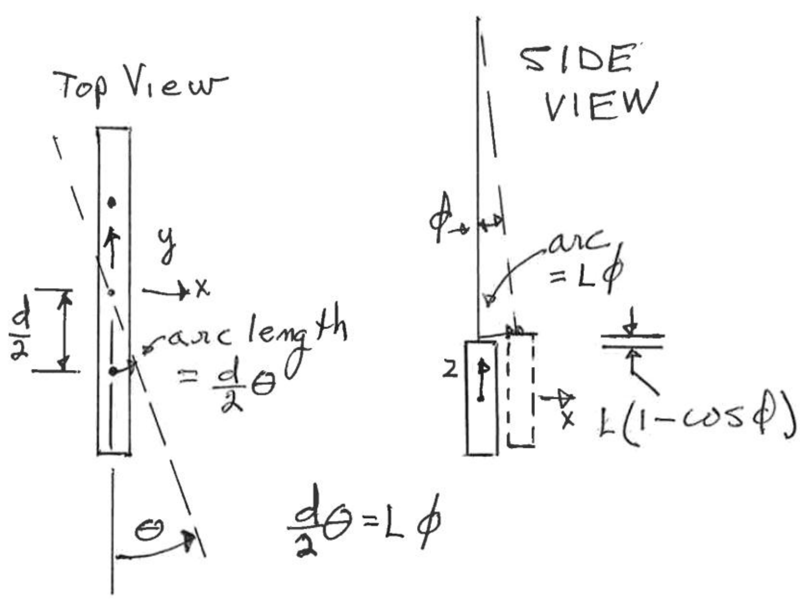


Solution:

a. A vertical axis passing through the center of mass is a principle axis of the object. For a rectangular uniform solid the mass moment of inertia with respect to a z axis passing vertically through G is given by:

$$I_{zz/G} = m\left(\frac{a^2}{12}+\frac{c^2}{12}\right).$$

b. To find the equation of motion for rotation about the z axis, we apply Euler's equation with respect to the center of mass. The geometry is critical to this problem. Consult the figure to see the relation between the angle the string makes to the block and the angle of rotation. If the block rotates through a small angle θ about the z axis, the string will make an angle ϕ with the vertical, such that $\frac{d}{2}\theta = L\phi$, as shown in the figures at right.



c. There is a tension in each string approximately equal to half the weight of the block mg/2. The component of tension perpendicular to the radius from G to the connection point creates a moment about the z axis.

$$\sum_{external} \vec{\tau}_O = I_{zz/G}\ddot{\theta}\hat{k} = 2 * \frac{d}{2}\hat{r} \times \left(-T\sin\phi\hat{\theta}\right) = -dT\sin\phi\hat{k} \bullet -dT\phi\hat{k} \text{ for small } \phi.$$

Since $\frac{d}{2}\theta = L\phi$, then $\phi = \frac{d}{2L}\theta$

$$\therefore \sum_{external} \vec{\tau}_O = I_{zz/G}\ddot{\theta}\hat{k} = -dT\frac{d}{2L}\theta = -\frac{d^2 T}{2L}\theta = -\frac{d^2 mg}{4L}\theta$$

$$I_{zz/G}\ddot{\theta}\hat{k} + \frac{d^2 mg}{4L}\theta = 0$$

$$\omega_n = \sqrt{\frac{\frac{d^2 mg}{4L}}{I_{zz/G}}} = \sqrt{\frac{\frac{d^2 mg}{4L}}{m(\frac{a^2}{12} + \frac{c^2}{12})}} = \sqrt{\frac{3d^2 g}{L(a^2 + c^2)}}$$

d. The potential energy in this problem is due to changes in gravity potential energy. As the block rotates through the angle θ about the z axis, the center of mass of the system moves in the positive z direction by the amount:

$z = L(1-\cos\phi) = L(1-\cos(\frac{d}{2L}\theta))$ where z is measured from the center of mass in its static equilibrium position at $\theta = 0$.

$V = mgL(1-\cos(\frac{d}{2L}\theta))$. The kinetic energy has two contributions. The first is the rotational KE about the center of mass. The second is the translational kinetic energy of the object. The only non-zero translational velocity of the center of mass is in the vertical direction.

$T = \frac{1}{2}I_{zz/G}\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2$ where

$$\dot{z} = \frac{d\left[L(1-\cos(\frac{d}{2L}\theta))\hat{k}\right]}{dt} = \frac{d}{2}\sin(\frac{d}{2L}\theta)\dot{\theta}\hat{k}$$

Therefore

$$T = \frac{1}{2}\left[\frac{m}{12}(a^2 + c^2)\right]\dot{\theta}^2 + \frac{1}{2}m\left[\frac{d}{2}\sin(\frac{d}{2L}\theta)\dot{\theta}\right]^2$$

The second kinetic energy term is a 4th order quantity involving the coordinate θ. For small angles of oscillation, this term is negligible compared to the rotational kinetic energy term and may be neglected. Since there is no damping considered in this problem the total energy of the system must be conserved, and its time derivative must be zero. The total energy of this single degree of freedom system is given by:

$$E = T + V = \frac{1}{2}\left[\frac{m}{12}(a^2 + c^2)\right]\dot{\theta}^2 + mgL(1 - \cos(\frac{d}{2L}\theta))$$

$$\frac{dE}{dt} = 0 = \left[\frac{m}{12}(a^2 + c^2)\right]\ddot{\theta}\dot{\theta} + mg\frac{d}{2}\sin(\frac{d}{2L}\theta)\dot{\theta}$$

$$\Rightarrow \left[\frac{m}{12}(a^2 + c^2)\right]\ddot{\theta} + mg\frac{d}{2}\sin(\frac{d}{2L}\theta) = 0$$

After linearization of the sine term the EOM is given by

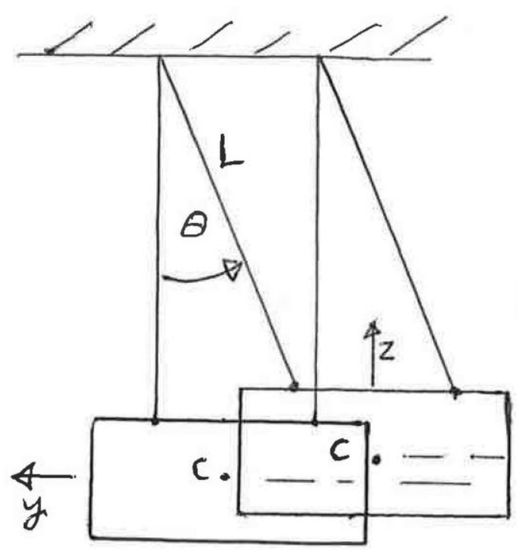
$$\left[\frac{m}{12}(a^2 + c^2)\right]\ddot{\theta} + mg\frac{d^2}{4L}\theta = 0$$

This is the same result as before.

Problem 7

Consider the same system as in problem 6.

a) Find the kinetic and potential energy expressions for the block when it swings in the plane of the paper as drawn in the figure. This means there is no torsional motion about the vertical axis passing through the center of mass. This is a one degree of freedom system.



b) Use the concept of conservation of total energy to find the equation of motion. Linearize the EOM and find the natural frequency.

c) Assume that we conduct an experiment with this system. The pendulum is given an initial angular deflection of 0.2 radians and released. Over time the oscillation amplitude becomes smaller due to damping. After 5 cycles of vibration the maximum angle of motion has reduced to 0.08 radians. What is the approximate damping ratio of the system.

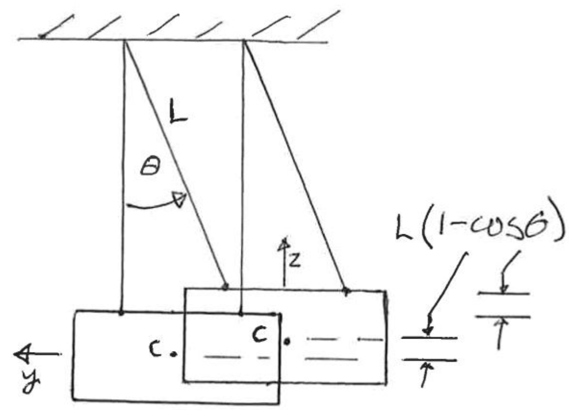
Solution:

a. When the block swings as drawn in the plane of the paper, it does not rotate. It only translates in the plane of the paper. See the diagram to the right. The total kinetic energy of the system may be accounted for by considering the velocity of the center of mass. The velocity of the center of mass is the same as the velocity at any other point on the block, including the point of attachment of either string. With this understanding it is more obvious to see that:

$$\vec{v}_{G/O} = \vec{\omega} \times \vec{r} = \dot{\theta}\hat{k} \times L\hat{r} = L\dot{\theta}\hat{\theta}$$

$$T = \frac{1}{2} m\vec{v}_{G/O} \bullet \vec{v}_{G/O} = \frac{1}{2} m\left[L\dot{\theta}\right]^2$$

$$V = mgL(1-\cos\theta)$$



The reference point for the potential energy was taken from the static equilibrium position of the center of mass of the block. Note that the attachment point of either string to the block moves upwards the same amount as any other point on the block, including the center of mass.

b. As in the previous problem there is no damping. Therefore the total energy of the system is conserved and the time derivative of the total energy of the system must be equal to zero. This will lead directly to the undamped equation of motion of the system:

$$\frac{d\left(T+V\right)}{dt} = \frac{d\left(\frac{1}{2} m\left[L\dot{\theta}\right]^2 + mgL(1-\cos\theta)\right)}{dt} = 0$$

$\Rightarrow mL\ddot{\theta} + mgL\sin\theta = 0,$ which when linearized for small angles yields

$$mL\ddot{\theta} + mgL\theta = 0$$

From the EOM it is now easy to see that the natural frequency is the same as that as a simple point mass pendulum of length L

$$\omega_n = \sqrt{\frac{g}{L}} \ .$$

c. The damping ratio for a single DOF system may be determined from the logarithmic decrement:

$$\zeta = \frac{1}{2\pi n}\ln(\frac{A_o}{A_n}) = \frac{1}{10\pi}\ln(\frac{0.2}{0.08}) = 0.029$$