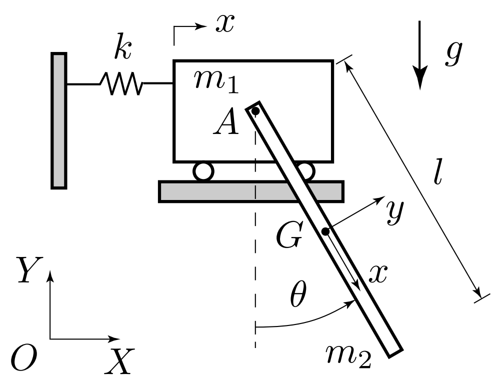
2.003SC
Recitation 8 Notes: Cart and Pendulum (Lagrange)

Cart and Pendulum - Problem Statement

A cart and pendulum, shown below, consists of a cart of mass, m_1 , moving on a horizontal surface, acted upon by a spring with spring constant k. From the cart is suspended a pendulum consisting of a uniform rod of length, l , and mass, m_2 , pivoting about point A .



Derive the equations of motion for this system **by Lagrange**. Specifically,

- Find T, the system's kinetic energy
- Find V, the system's potential energy
- Find ${v_G}^2$, the square of the magnitude of the pendulum's center of gravity

Cart and Pendulum - Solution

Generalized Coordinates $q_1 = x, \qquad q_2 = \theta$

Kinematics

The linear velocity of the pendulum's center of mass, v_G , is given by

$$v_G = \dot{x}\hat{I} + \frac{l\dot{\theta}}{2}\hat{j} = (\dot{x} + \frac{l\dot{\theta}}{2}cos\theta)\hat{I} + (\frac{l\dot{\theta}}{2}sin\theta)\hat{J}$$

The square of its magnitude is given by

$$v_G^2 = (\dot{x} + \frac{l\dot{\theta}}{2}cos\theta)^2 + (\frac{l\dot{\theta}}{2}sin\theta)^2$$

Expanding and simplifying,

$$v_G^2 = \dot{x}^2 + 2\frac{l\dot{x}\dot{\theta}}{2}cos\theta + (\frac{l\dot{\theta}}{2})^2cos^2\theta + (\frac{l\dot{\theta}}{2})^2sin^2\theta$$

$$v_G^2 = \dot{x}^2 + \dot{x}\dot{\theta}lcos\theta + \frac{l^2\dot{\theta}^2}{4}$$

Kinetic Energy

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2{v_G}^2 + \frac{1}{2}I_G\dot{\theta}^2$$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + \dot{x}\dot{\theta}lcos\theta + \frac{l^2\dot{\theta}^2}{4}) + \frac{1}{2}(\frac{ml^2}{12})\dot{\theta}^2$$

Potential Energy

$$V = \frac{1}{2}kx^2 + m_2g\frac{l}{2}(1 - cos\theta)$$

Lagrangian

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + \dot{x}\dot{\theta}lcos\theta + \frac{l^2\dot{\theta}^2}{4}) + \frac{1}{2}(\frac{ml^2}{12})\dot{\theta}^2 - \frac{1}{2}kx^2 - m_2g\frac{l}{2}(1 - cos\theta)$$

Lagrange's equation for the first generalized coordinate,

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{x}}) - \frac{\partial\mathcal{L}}{\partial x} = Q_x$$

yields the first equation of motion.

$$(m_1 + m_2)\ddot{x} + \frac{m_2l}{2}\ddot{\theta}cos\theta - \frac{m_2l}{2}\dot{\theta}^2sin\theta + kx = 0 \tag{1}$$

Lagrange's equation for the second generalized coordinate,

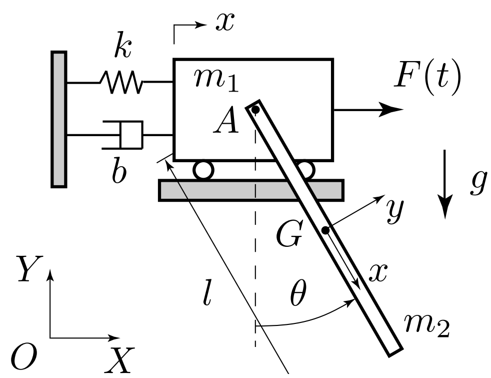
$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}) - \frac{\partial\mathcal{L}}{\partial\theta} = Q_\theta$$

yields the second equation of motion.

$$(\frac{m_2l^2}{4} + \frac{m_2l^2}{12})\ddot{\theta} + \frac{m_2l}{2}\ddot{x}cos\theta + m_2g\frac{l}{2}sin\theta = 0 \tag{2}$$

Cart and Pendulum - Problem Statement

Assume that the cart and pendulum system now contain a damper/dashpot of constant b between the cart and ground, as well as an external force, $F(t)$, applied to the cart.



Derive the equations of motion for this system **by Lagrange**. Specifically, show the generalized forces.

Cart and Pendulum - Solution

Both the damper and the external force are non-conservative (forces). Consequently, they enter the Lagrange formulation as generalized forces.

Generalized Forces

$$\delta\mathcal{W}^{nc} = \sum_{i}^{N} \mathbf{f}_i^{nc} \cdot \delta\mathbf{R}_i = \sum_{j=1}^{n} \mathbf{Q}_j \delta\xi_j$$

$$\delta\mathcal{W}^{nc} = [-b\dot{x} + F(t)]\delta x + [0]\delta\theta$$

$$Q_x = -b\dot{x} + F(t) \qquad\qquad Q_\theta = 0$$

which changes the first equation to become,

$$(m_1 + m_2)\ddot{x} + \frac{m_2 l}{2}\ddot{\theta}cos\theta - \frac{m_2 l}{2}\dot{\theta}^2 sin\theta + kx = -b\dot{x} + F(t)$$

So our equations of motion become

$$(m_1 + m_2)\ddot{x} + \frac{m_2 l}{2}\ddot{\theta}cos\theta - \frac{m_2 l}{2}\dot{\theta}^2 sin\theta + b\dot{x} + kx = F(t) \tag{3}$$

and

$$(\frac{m_2 l^2}{4} + \frac{m_2 l^2}{12})\ddot{\theta} + \frac{m_2 l}{2}\ddot{x}cos\theta + m_2 g\frac{l}{2}sin\theta = 0 \tag{4}$$