PROFESSOR: Right. And we've done some review. Something else.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, symmetry, review principal axes ideas. What else? I think we spent quite a bit of time on generalized forces and ways for computing them. All right. And I'm trying to begin to summarize what we've done up to this point because we have a quiz coming next Tuesday. So some of yesterday's lecture was intended to kind of start pulling together, comparing direct versus Lagrange, advantages, disadvantages. And on Tuesday next, we'll do more examples.

OK, so the problem you're going to work on today-- and you really are going to work in groups. And how many do we have here? We've got four-- about enough for three groups, let's say. We have a double pendulum. It's made out of two-- now I have two pieces of chalk. It's two slender rods, but an approximation-- this is a double pendulum.

They have many-- that one and this one have a lot in common. Takes how many coordinates to completely describe the motion? One for each rod, right? And the coordinates up there that are drawn in that diagram is the angle of the first one with the vertical and the angle of the second one with a vertical. And with that, you can completely describe any allowable motion of the system.

So a double pendulum has a property that it's got two natural frequencies and two mode shapes. That's the shape of the first mode. Both masses go in the same direction. And the second mode looks like that. The two masses go in opposite directions. Not with equal amplitudes, but opposite directions. And it's a different frequency too. It's higher.

So it has two natural frequencies, two mode shapes. And if you had-- in this case it shows a force acting on the system, pushing this thing back and forth. And we're interested in the generalized forces, or thinking in terms of the work done by the non-conservative forces.

So the exercise to do in groups-- like four or five of you are a group. You five are a group. And you two, four, five are a group there. Work this out. Find this vector. This is the position vector to the point of application of the force. And we want you to do that. Actually, I did something last time, and I'm almost forgetting to do it.

I want to remind you of something. Professor Gossard, each week for the recitations-- he teaches Thursday ones-- does a write-up of the recitation. And they're posted. So this week the solution to this problem is posted. But also, a little quick review of the important stuff. So this would be posted. And basically I've redrawn it up here on the board.

So this is this kinematic method for obtaining generalized forces. So a body with N forces on it. Here is the ith one. At each point of application of the force, there is a total virtual displacement that basically comes from the sum of all the individual generalized coordinate virtual motions. You sum them up, you get the total.

And so therefore, at the point of application of each force, there is a total amount of virtual work that's done. So the total non-conservative work is the sum of the virtual work done at the point of application of every force summed over all the forces. And remember, inside of here, the total displacement at every location is a summation over of all the generalized coordinates. So it really ends up as a double sum, this dot product, the forces times delta ri. It's a double summation over the forces that are applied and over the generalized coordinates.

But at the end of the day, what you end up doing, the calculation you want to do, is you need to know the generalized force-- the force associated with each generalized coordinate. And that's the summation of the contribution of the virtual work caused by that coordinate's motion dotted into the force at every location. So at every location, there's a little contribution to the total work done. And therefore, the total generalized force is this sum.

So today's problem is easy in the sense that i is 1. You only have to deal with one force. But this is a little messier problem in that in order to do this, you need to be able-- this is the r sub i in this case. Here's point P. I want you to first, just in groups, work out what this vector is, rP, in the system as drawn. Here's the inertial system xy, generalized coordinates theta 1, theta 2.

And this bar is L1 long, and this bar is L2 long. And come up with an answer in a form like this, something in the i hat direction plus something in the j hat direction. And then as your group finishes-- this is group one here-- come put your answer up. Second group, put your answer up. Third group, put your answer up. And we'll move on. This will go pretty fast.

All right, looks like everybody's pretty much in agreement. Not much to talk about. So let's do the next piece. So the next part is to compute your variations, this part of the calculation, for each of the coordinates. So do rP with respect to theta 1 and rP with respect to theta 2 and write down your answers. So the last step here is get q theta 1 q theta 2. So move on to getting the two generalized forces now.

AUDIENCE: [INAUDIBLE].

PROFESSOR: You could. And then if you did, then this would have to be a summation over two components. Remember, this is a sum. It just happens in this example, we only have i goes from 1 to 1. But if there were two forces, then you would do this twice. You'd have an r1 with respect to theta 1 and an r2 with respect to theta 1. And you'd do this product twice, add the two pieces together, to get the total-- the total generalized force is the sum of the bits that come from all of the individual forces.

AUDIENCE: So if [INAUDIBLE] just doing theta 1, but [INAUDIBLE] two forces, you just do one force, and then you do [INAUDIBLE] force--

PROFESSOR: Yeah, so now here let's have a B. We'll call it just B force. And it has to Bx components and By components. You would now also have to compute what would be rB. You would find it. Then you'd have rB would be this vector, which is just that piece plus this piece, right? So you'd have just those-- rB would be this plus this, and then you would do the derivative of rB with respect to theta 1. And you'd get something. And derivative of rB with respect to theta 2.

AUDIENCE: So then you'd just do the rB equals [INAUDIBLE] for one force, and then you do the rPe [INAUDIBLE]?

PROFESSOR: Yeah. So you have two-- so this thing would end up being the summation of two contributions, right? This would look like an FB dot rB with respect to-- and this is just for just one of them. So we'll do the theta 1. With respect to theta 1 plus F-- I'm just calling this one F-- dot derivative rP with respect to theta 1. So you get two contributions from the two forces give you the total q. This would be q theta 1. And then you'd do it again for the theta 2. And you'd have two possible pieces.

AUDIENCE: So you try to find r's for every force [INAUDIBLE].

PROFESSOR: At every point of application of a force, you define an r. You eventually need to do this summation for every force. All right. Everybody's got the same answers. So now I'm going to try to-- this is an indirect way of answering your question. I'm now going to try to convince you that you're wrong. And I want you to tell me why my logic is flawed.

AUDIENCE: No.

PROFESSOR: OK? So I look at these answers here, and so this says that if-- this line right here, if I caused a little delta theta 1 here, that swings this arm through that amount. And it moves a distance in the x direction, L1 cosine theta delta theta 1. And it moves over that little bit right here, right? OK. And you're telling me that this point over here moves that amount.

The point of application of this force due to delta x, delta theta 1, is this delta theta 1. There's no L2 involved. So to me, it seems like here's this pendulum. And if I move this first piece by a little bit, delta theta 1, this piece also moves. And so if you just think of a straight piece, delta theta 1 times L1 moves a little bit. But down here it moves like twice as much. So I think that when you move this little delta theta one, that you ought to get even more motion down here. And yet your answers say that that's not true. Tell me why-- what's flawed in my reasoning?

AUDIENCE: So when you-- I'll try. When you [INAUDIBLE] theta 2, [INAUDIBLE]. And now when you move by theta 1, [INAUDIBLE] all points move the same theta. So even though [INAUDIBLE].

PROFESSOR: OK. So you're saying that we argue that when you have one virtual displacement, or in this case a rotation delta theta 1, we freeze all other generalized coordinates. That means we freeze delta theta 2 and don't let it change. Delta theta 2 is measured with respect to the vertical, right? So if that moves over delta theta 1, this whole thing would-- this angle here we go from delta theta 2 to delta theta 2 plus delta theta 1 if my argument's correct. But for my argument to be correct, that would have to swing by delta theta 1 over here as well.

But the fact that we freeze theta 2, in fact, the angle between these two things has to change. This has to open up a little bit. So when this is moving over, this is dropping down so that this point, this whole body only translates. So that whoever described it as just pure translation of the second body, that's another way of saying it. So you've got to remember what it means, freezing all the other generalized coordinates and only allowing the one you picked to move.

Good. All right. Nice work. We've got some time left. So do you have any questions from problem sets, lectures, just--

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah.

AUDIENCE: When you're calculating kinetic energy of a rotating body, there's an omega [INAUDIBLE]. In the homework, the last homework, I didn't know which omega could take, whether to take it from the wheel or from the axis.

PROFESSOR: Oh, this is the rotating thing with the--

AUDIENCE: Yes.

PROFESSOR: Ah, OK.

AUDIENCE: There's two omegas there, and I didn't know which omega [INAUDIBLE].

PROFESSOR: So should we just talk about that problem? Are other people-- this is the one where you had basically the rod, and on the end of the rod was the wheel, which was rotating. So this was going around at some omega 1 in the z, right? And this was rotating, and I think this was like z, maybe x and y. So it was rotating about that.

AUDIENCE: [INAUDIBLE] it had a negative. It was a negative [INAUDIBLE].

PROFESSOR: Oh, it was a-- yeah. Actually, I think I have it here with its correct definition. So rather than try to--

AUDIENCE: Yeah, because there was the positive [INAUDIBLE], and then there's the [INAUDIBLE].

PROFESSOR: Here it is. Let's get the picture right. And there was an x, a y, and a z. And this was cap omega. And this was omega 1. And that's minus, in minus direction, for this system, right? OK. So then we're trying to compute T for this system. 1/2, and what'd we call for the masses here? This was m1, and this was m2, I guess. Yeah.

So this T for 2-- T1 you don't have a problem with, right? T1 is just a shaft, just a rod. It's pivoting about its end 1/2i about-- if this is-- I call it o. 1/2i, i with respect to o, mL squared over 3 times that squared. You're done. That's the energy of this part. This is the problematic one, right? The piece of it. So 1/2, I would say mass 2. And this does have a point. They called this point B here. Velocity of B dot velocity of B.

So that's the 1/2mv squared piece of it. Plus 1/2 omega dot h with respect to G for-- this is mass 2. So you have a contribution to the kinetic energy of this that comes from its center of gravity translating, and a contribution from the rotor rotating with respect to its center of mass here, G here. And that'd be the expression you use.

And then the problem is, what omega is the trick here, right?

OK. This is the one that had the concept question I talked about in class because it came up, and I hadn't thought about this, about the body coordinates attached to this are actually rotating with this. And that gets messy trying to figure out what are the-- how do you break-- you're supposed to express the angular rotation in the body coordinates. But this angular rotation, this one actually is lined up with this axis.

And you could have body coordinates on here. You could have a-- if this is a y, you could have a body coordinate x and z that rotate with it. But this Component would just be still along this axis, right? So the concept question [INAUDIBLE] was basically, are these body coordinates, x, y, z, attached to the rod?

Principal axes for this body. And the complication here is that this body is rotating. And so this x and this z don't rotate with that body. And so they're not body fixed coordinates for that body. But in fact, they're still principal coordinates for that body. Why is that true?

AUDIENCE: Because you have [INAUDIBLE]. And you can take whatever you want for the [INAUDIBLE].

PROFESSOR: You're close. This body is axially symmetric. And so at any instant in time, you could say that-- you could define a principal coordinate on this body that is lined up with these. So for axially symmetric bodies, you could do that. You can do that. You can just say, at an incident in time, let's compute the kinetic energy.

And we have the rotation of that body defined in the instantaneous principal coordinates of the body. So now omega for the second body is this in the k minus omega 1 in the j of the x, y, z system rotating with the rod. And then to complete this, what's H? So H with respect to G is Ixx Iyy Izz times-- and now this one's rotation is 0 minus omega 1 and cap omega.

And I multiply that times this with the definition that the first one gives me the I, second one J. And so I end up with two pieces. The first one is zero. I get an Iyy.

And remember, these are defined with respect to center of mass of the disk. Iyy minus Iyy omega 1 j plus Izz cap omega k.

And that'd be the angular momentum for the disk. Right? And then the kinetic energy, the 1/2 omega dot H, you now just have another omega 1/2, and you have a vector, 0 minus omega 1 J cap omega k dotted with the H vector, which is your 0 Iyy omega 1 j and Izz cap omega k.

AUDIENCE: I have a question. [INAUDIBLE].

PROFESSOR: Yeah. And if that one's negative, then it'll happily fix this one because it ought to come out looking like omega squareds, right? And so out of this, you get Iyy omega 1 squared plus Izz cap omega squared. And that's the second piece, and you add it to this piece.

AUDIENCE: So is there ever a time where the omega's inside [INAUDIBLE].

PROFESSOR: So let me-- we have a total-- so let's just talk-- we have H with respect to G. Or a slightly simpler example.

So H for this system with respect to the center is H with respect to G plus rG with respect to the center cross PG, right? I can write the angular momentum that way. And first piece we know. The second piece, rG, is in a r hat direction, if you will. The P is going around the circle.

And so the cross product of those two gives me a component, what's the direction of the result. So this is r cross theta hat. You get a k. This ends up being-- this gives you a k component. And this one we write out. So this is minus Iyy omega 1 j plus Izz cap omega k plus some stuff that's in the k. Agreed?

And I want to know the torques required to make this system do what it's doing. How do I get the torques required to make the system go around?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah. Sum of the external torques equals dH. Now, with respect-- remember, we would ordinarily usually write that as an A. We'll call it H with respect to O dt plus velocity-- usually we call it A-- with respect to O, cross P. Right? In this system, what's the velocity of this piece, this term? Zero. So we don't have to worry about it. So the sum of the external torques should just be the time derivative of this angular momentum vector. Right?

And dH then O dt is equal to the partial derivative with respect to t of H. And I'll write it like this. This is in the rotating frame. So this is the piece from inside the frame plus, this is just the derivative of a rotating vector. Omega cross H. And the issue here is, what's the omega? That's really what this question comes down to.

When we did this problem, we found that H came from taking the omega, the vector, multiplying it by this, and we got these two pieces. And so it started off with components in the j and k direction, and it came out with components in the j and k because this was diagonal. So we have principal axes. We came out with these two pieces. So the angular momentum has a j component and a k component.

What is actually-- now, this vector, this angular momentum vector, the reason we have this second term is because of the change of direction of a vector. This piece comes from taking the time derivatives of the unit vectors in the problem. The time derivatives of the other stuff in the problem has been taken care of by this. So this only deals with the time derivatives of the unit vectors.

So what is the actual rotation rate in this problem, the rate at which things are changing direction? Which one's changing direction?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Is this one changing direction? No. So actually, it doesn't give you any derivative, right? Is this one changing direction? Yeah. And at what rotation rate is it changing direction?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Capital omega. But why not-- but it's not changing little omega, is it? The math, the vectors works that all out for you. Because this is in the j direction, and this is the same. You just use the same total rotation vector here. And this will have in it the zero, the omega minus omega 1 j, the plus cap omega k. And the parts that you don't care-- this crossed with itself goes to zero.

So even though you're just going ahead and leaving this in here, it doesn't result in anything because the cross product would-- cross product with itself gives you nothing. So the only piece that actually contributes nonzero contribution to this answer is this crossed with the pieces in here. So when you're taking derivatives of rotating vectors, remember the original formula.

And the original formula is the derivative of the vector in the rotating frame-- is this magnitude getting more or less-- plus just the rotation rate crossed with the original vector. And the pieces that are common just fall out. The piece of this that is due to its own rotation doesn't enter into it. This rotation doesn't contribute to that d by dt of the rotation. Just works out for you. Yeah.

AUDIENCE: What would happen if that just [INAUDIBLE] rotates like this, but also rotates on its own axis as it does it?

PROFESSOR: You mean if this one is going in the k?

AUDIENCE: Yeah.

PROFESSOR: Then--

AUDIENCE: [INAUDIBLE] you didn't have the big omega term because the actual thing isn't-- the big omega would be taken into account when looking at the first term of kinetic energy.

PROFESSOR: If this one also had some k rotation in addition relative to here, we'll call it omega 2 in the k direction, then what's the total rotation rate in the k direction for the disk?

AUDIENCE: [INAUDIBLE].

PROFESSOR: You add cap omega, 2 omega 2, and you'd have its total k directed rotation, right?

And if you did that, you'd end up in here with an omega plus omega-- cap omega plus omega 2 k. But when you came over to do the time derivative, you just leave all those things in there. And the ones that you don't care about will just cancel out because they're cross products with themselves.

AUDIENCE: [INAUDIBLE] omega also appears [INAUDIBLE].

PROFESSOR: Oh yeah. Well, the cap omega appears in this piece. But this goes to k. We did a particular problem. I'm not trying to prove this in general, although there's probably a way to do that. This problem, this term is in the k direction. And therefore, k hat does not change direction with time. Therefore, the dH dt of this piece goes to zero. So we only had to deal with this part in taking the time derivative that dealt with changes in direction. It's only the changes in direction that the omega cross something matters, right?

Good question. Something else? Recent problems. Somebody in the last class asked about, had a question about this thing. That bring anything to memory?

AUDIENCE: [INAUDIBLE].

PROFESSOR: I see some grimaces. What troubled you about that problem? What troubled the last group about that problem was the posted solution. They didn't understand the posted solution. So I ran through-- I showed why the posted solution works the way it does.

AUDIENCE: Well, one confusion that I had had doing this problem was that the spring for the wheel mass, when that exerts displacements, Newton's third being what it is, should it also exert some force on the large base mass as well, so you should take that into account for the total displacemenet caused by springs for big mass, the big--

PROFESSOR: Are you talking about not computing-- just giving equations of motion, or computing generalized forces, or what's the context?

AUDIENCE: Yes, when you're computing your equations of motion, you have to take into account from the larger mass of the displacements in the spring, it's attached to the wall, [INAUDIBLE].

PROFESSOR: As well as the-- yeah. So if you're doing this by the direct method, you have to figure that out. So here's a way of doing it. This is a way that I think the-- the way the student just read off his paper. So I said, let's talk about the forces in the x direction on the main mass. Well, that's mass 1 times its acceleration must be equal to all the external forces. This is minus kx minus bx dot. This is minus the x component of the spring force minus the x component of the friction force from the wheel minus the x component of any normal force from the wheel.

And I do the same thing for the second mass, M2. The total forces on it must be equal to its mass times its acceleration. Well, its acceleration I'm doing this just to get in the x direction. Its acceleration in the x is the main mass, x double dot, which it's sitting on. And the x component of the relative coordinate x1 double dot. This is the total acceleration in the x direction of the second mass.

And the total forces on that second mass are-- signs are changed. Internal force, internal force, internal force, plus the force that was applied. Actually, here it is written out. Here is that external force that was applied to that wheel. If you take these two equations, you just add them together. All of the internal forces drop out.

Plus fx minus fx. All of those drop out. And you end up with an expression that says, this is a system. The total external forces on the system in the x direction must be equal to the total mass of the system times the acceleration of its center of mass. Right? That's what Newton said.

Well, the acceleration of the center of mass is the sum of the individual pieces times their individual accelerations. The sum of the total mass times the acceleration of the center of mass is equal to the sum of the individual masses times their individual accelerations. Mass one, its acceleration. Mass two, its acceleration due to its local coordinate plus its acceleration due to the main mass acceleration. And all of that's got to be equal to the external forces, which are now only minus kx, minus bx dot, and plus F.

And this term here, this stuff, all that, including this, these three M X double dot terms, all added together as the same thing is the total mass times the acceleration of the center of mass, which is somewhere. But that's a completely legitimate equation for this system. And it involves both masses, both coordinates. So this one here would just be some x1 double dot times a cosine phi or a sine phi or something like that.

AUDIENCE: So for this problem, is there only one equation of motion?

PROFESSOR: Oh no. There still has to be two. But I've just shown you a way of getting at one of them without ever solving for the internal forces. By just realizing that if I separate them, that Newton's third law, all the internal ones will cancel when I add the two pieces together. I still have to get a second equation. Where would you get it?

AUDIENCE: Sum of the forces in x1?

AUDIENCE: They're rotating [INAUDIBLE].

PROFESSOR: Yeah, you want to do something where you don't have to solve-- you don't want to have to solve for any of these internal forces that you don't know, right? So the spring force you can know because it is a minus k times the coordinate that you have to use, minus kx1. So I would go in here, and I'd say maybe the-- what's the sum of the torques about that point, perhaps.

AUDIENCE: There's no slip, right?

PROFESSOR: And no slip. So these things-- that avoids having to deal with any of these forces. The real force acting on it, real force, you have to deal with that. And the minus kx1 force you need. It comes into your equation of motion. If you do that particular approach, it has one problematic-- it's not terrible but you have to remember to deal with it-- is that the sum of the x torques about that point we'll call A is d HA dt plus vAo cross P, right?

Or I gave you an equation yesterday. This is easier to do usually if you do d HG dt plus rG with respect to A cross the mass times the acceleration of g with respect to

o. This would be the second mass, M2. You can derive. It's about three lines. You can prove that this statement's the same as that. This one, turns out this one's less work. And the reason that this one's more work is this term always cancels out.

Cancels out with a piece of it is generated when you do this calculation. But you know how messy this stuff gets. You've got all these unit vectors running around and cosine thetas and sine thetas. It's a lot of work to get this piece. And it's twice the work because you have to go in and find the piece that cancels it in here. So if you actually-- I sat down about a week ago and said, there's got to be a way to get rid of this thing.

And there is, and it comes out like this. Now there's no pieces that cancel. You cancelled out in the proof, and get this piece to go away. And you're left with this. So this is the distance from your point to the center of mass. In this case, it's just r. And the only thing you have to calculate is the acceleration of the center of mass, which is capital X double dot plus X1 double dot. You know that. So this is vastly easier to do.

AUDIENCE: So I understand how to use that if I'm using different [INAUDIBLE].

PROFESSOR: Center of mass, I didn't get the last bit.

AUDIENCE: [INAUDIBLE].

PROFESSOR: That's the other. Because you're doing it around point A is what gets you into this mess. If you're doing it-- as soon as you work with respect-- if you're working with G, the center of mass, then this is vG cross vG, which is always zero. This has a vG in it. So that's the reason we like to work around center of the mass, but we don't do centers of mass when it causes us to have a whole bunch of external things we don't know. Thanks. Yes.

AUDIENCE: [INAUDIBLE].