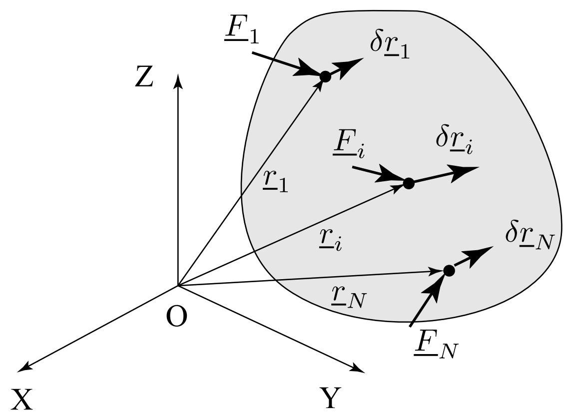
2.003SC
Recitation 9 Notes: Generalized Forces with Double Pendulum Example

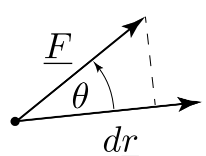
Generalized Forces

Consider a system with n generalized coordinates, q_j , acted upon by N non-conservative forces, where

- $\underline{F}_i$ = non-conservative forces acting on the system
- $\underline{r}_i$ = positions of the points at which the non-conservative forces act
- $\delta\underline{r}_i$ = virtual displacements of those points when generalized coordinates are varied



Recall that, when a force goes through a displacement,



the work done is given by $\mathcal{W} = \underline{F} \cdot d\underline{r} = |\underline{F}||d\underline{r}|cos\theta$

since only the component of the force in the direction of the motion "counts".

So the virtual work done by all the non-conservative forces is

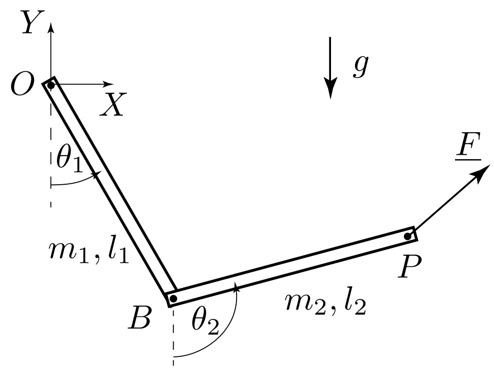
$$\delta\mathcal{W}^{nc} = \sum_{i=1}^{N} \underline{F}_i \cdot \delta\underline{r}_i = \sum_{i=1}^{N} \underline{F}_i \cdot \sum_{j=1}^{n} \frac{\partial \underline{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} (\sum_{i=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j})\delta q_j = \sum_{j=1}^{n} Q_j \delta q_j$$

And the generalized forces are given by

$$Q_j = \sum_{i=1}^{N} \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \qquad j = 1, 2, ..., n$$

Generalized Forces on Double Pendulum - Problem Statement

A double pendulum is shown below with a force $\underline{F}$ acting at point P. The generalized coordinates are $q_1 = \theta_1$ and $q_2 = \theta_2$.



Determine the generalized forces for the system.

Intermediate steps:

- Write an expression for $\underline{r}_P$, (the location of the point at which the force is applied).
- Write expressions for $\frac{\partial \underline{r}_i}{\partial \theta_1}$ and $\frac{\partial \underline{r}_i}{\partial \theta_2}$.
- Write an expression for the generalized forces, Q_{θ_1} and Q_{θ_2}.
- Write an expression for the virtual work force $\underline{F}$ does on the system.

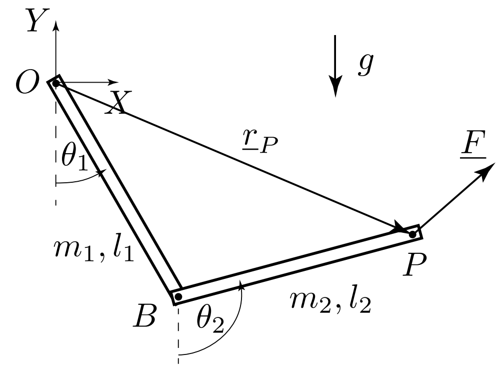
Recall that the generalized forces are defined by the following

$$Q_j = \sum_{i=1}^{N} \underline{f_i} \cdot \frac{\partial \underline{r}_i}{\partial q_j} \qquad\qquad j = 1, 2, ..., n$$

where

- n = number generalized coordinates, generalized forces
- N = number of forces acting on the system
- $\underline{f}_i$ = forces acting on the system
- $\underline{r}_i$ = location of points at which forces act
- q_j = the system's generalized coordinates

Generalized Forces on Double Pendulum - Solution



$\underline{r}_P$, which describes the location of point P , can be expressed as

$$\underline{r}_P = (l_1 sin\theta_1 + l_2 sin\theta_2)\hat{I} + (-l_1 cos\theta_1 - l_2 cos\theta_2)\hat{J}$$

So the variations are given by

$$\frac{\partial \underline{r}_i}{\partial \theta_1} = l_1 cos\theta_1 \hat{I} + l_1 sin\theta_1 \hat{J}$$

$$\frac{\partial \underline{r}_i}{\partial \theta_2} = l_2 cos\theta_2 \hat{I} + l_2 sin\theta_2 \hat{J}$$

The non-conservative force, $\underline{F}$ can be expressed as follows.

$$\underline{F} = F_x \hat{I} + F_y \hat{J}$$

So the generalized forces are

$$Q_{\theta_1} = \underline{F} \cdot \frac{\partial \underline{r}_i}{\partial \theta_1} = F_x l_1 cos\theta_1 + F_y l_1 sin\theta_1$$

$$Q_{\theta_2} = \underline{F} \cdot \frac{\partial \underline{r}_i}{\partial \theta_2} = F_x l_2 cos\theta_2 + F_y l_2 sin\theta_2$$

The virtual work done on the system is

$$\delta \mathcal{W}^{nc} = Q_{\theta_1} \delta\theta_1 + Q_{\theta_2} \delta\theta_2$$

$$\delta \mathcal{W}^{nc} = (F_x l_1 cos\theta_1 + F_y l_1 sin\theta_1)\delta\theta_1 + (F_x l_2 cos\theta_2 + F_y l_2 sin\theta_2)\delta\theta_2$$