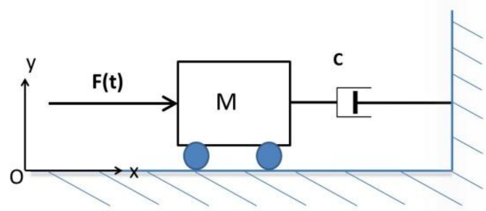
2.003 Engineering Dynamics
Problem Set 7--Solution

Problem 1:

A cart of mass'M' is connected to a dashpot with constant 'c'. A force F(t) acts on the mass, where the force is given by $F(t) = F_o \cos(\omega t)$.



a). Use Lagrange's equations to derive the equation(s) of motion for the system.

Concept question: A long time after applying F(t) the mass will exhibit what kind of motion? a). oscillate and move steadily to the right, b). oscillate and move steadily to the left, c). oscillate about a mean position, d). not move.

Solution: Specify coordinates: This is a one degree of freedom system. The coordinate x, is a complete, independent and holonomic set of generalized coordinates for the problem.

Find T and V: The potential energy V=0 and does not change.

The kinetic energy T comes purely from translation: $T = \frac{1}{2} m\dot{x}^2$.

Find the generalized forces: Generalized forces arise from the applied external force F(t) and the damper force F_d. In general one can define a position vector $\mathbf{r_i}$ which accounts for the motion of the rigid body at each point of application of an external non-conservative force. In general $\mathbf{r_i}$ is a function of all of the independent generalized coordinates used in the problem. One at a time we need to compute the virtual work done by a small virtual displacement of each generalized coordinate. This may be done intuitively for simple problems, such as this one. Since there is only one generalized coordinate x and both the damper force and F(t) are in the same direction as x then the virtual work done in a small variation δx would be given by $Q_x \delta x = \left[F_o \cos(\omega t) - b\dot{x}\right]\delta x$ and therefore the generalized force in this problem is simpley $Q_x = \left[F_o \cos(\omega t) - b\dot{x}\right]$.

More complicated problems require the use of a vector mathematics approach. This approach is applied to this simple problem as a model for more difficult problems later.

The explicit, exact, vector math approach:

$\vec{r}_i = f(x) = x\hat{i}$

$\frac{\delta \vec{r}_i}{\delta x}\delta x = \frac{\delta f(x)}{\delta x}\delta x = \frac{\delta (x\hat{i})}{\delta x}\delta x = \hat{i}\,\delta x$

$Q_x \delta x = \sum \vec{F}_i \bullet \frac{\delta \vec{r}_i}{\delta x}\delta x = \hat{i}\,\delta x \bullet (F_o \cos(\omega t)\hat{i} - b\dot{x}\hat{i}) = \left(F_o \cos(\omega t) - b\dot{x}\right)\delta x$

$\Rightarrow Q_x = F_o \cos(\omega t) - b\dot{x}$

Apply Lagrange's equations:

a) Find the equations of motion using Lagrange equations, which may be stated as:

$\frac{d}{dt}\left(\frac{\partial L}{\partial q_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$ where $L=T-V$. For purely mechanical systems that have only springs and gravity as potential forces, the Lagrange equations may be expressed in a much simpler form, which saves many steps in carrying out the math. It is valid any time that potential energy is only from gravity and springs.

$\frac{d}{dt}\left(\frac{\partial (T)}{\partial \dot{q}_j}\right) - \frac{\partial (T)}{\partial q_j} + \frac{\partial (V)}{\partial q_j} = Q_j$, where the q_j are the generalized coordinates. This equation must be applied once for each generalized coordinate in the problem. In this problem, there is only one generalized coordinate, x. T and V are functions of x and $\dot{x}$. Numbering the terms on the LHS as 1, 2 and 3 in order of appearance, the three terms may be evaluated in a systematic way.

Term 1: $\frac{d}{dt}\left(\frac{\partial (T)}{\partial \dot{x}}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x}$

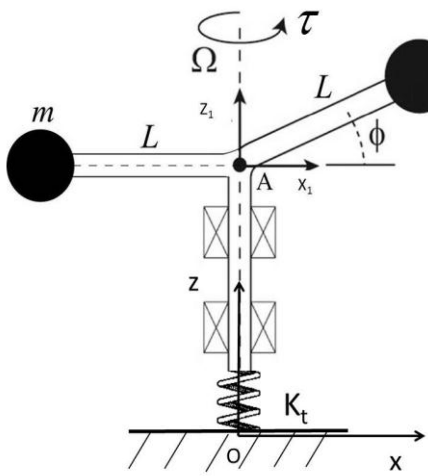
Term 2: $\frac{\partial (T)}{\partial x} = 0$

Term 3: $\frac{\partial (V)}{\partial x} = 0$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_x found above yields:
$$m\ddot{x} = Q_x = F_o \cos(\omega t) - b\dot{x}$$
$$\Rightarrow m\ddot{x} + b\dot{x} = F_o \cos(\omega t)$$
. This is a second order linear differential equation which may be solved. After initial transients have died out the steady state particular solution will be in the form of $x(t) = x_o \cos(\omega t - \phi)$. This is harmonic motion about some mean position which will depend on initial conditions, which is the answer to the concept question.

Problem 2:

Two identical masses are attached to the end of massless rigid arms as shown in the figure. The vertical portion of the rod is held in place by bearings that prevent vertical motion, but allow the shaft to rotate without friction. A torsional spring with spring constant K_t resists rotation of the vertical shaft. The shaft rotates with a time varying angular velocity Ω with respect to the O_{xyz} inertial frame. The arms are of length L. The frame A_{x1y1z1} rotates with the arms and attached masses. A time varying torque is applied to the rotor about the z axis, such that $\tau_{/A} = \tau_o(t)\hat{k}$. Note that the angle ϕ is fixed.



a). Derive the equations of motion using Lagrange equations.

Concept question: What are the units of the generalized force in this problem. a). N, b). N/m, c). N-m, d. other.

Solution: Specify coordinates: There is one degree of freedom. The coordinate $\theta(t)$, which is rotation about the z axis, makes a complete, independent and holonomic set of generalized coordinates for the system. By this choice of coordinate, recognize that $\vec{\dot{\theta}}(t) = \Omega\hat{k}$.

Find T and V: These were found in Pset 6. They are:

$$T = \frac{1}{2}\vec{\omega}\bullet\vec{H}_{/A} = \frac{1}{2}\begin{bmatrix} 0 & 0 & \Omega\hat{k}_1 \end{bmatrix}\bullet\left\{\vec{H}_{/A}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 0 & \Omega\hat{k}_1 \end{bmatrix}\begin{Bmatrix} -mL^2\Omega\sin\phi\cos\phi\hat{i}_1 \\ 0 \\ mL^2\Omega(1+\cos^2\phi)\hat{k}_1 \end{Bmatrix} = \frac{1}{2}mL^2\Omega^2(1+\cos^2\phi)$$

The potential energy comes from the torsional spring. The zero spring torque position of the rotor is selected as the reference position for V=0. At static equilibrium, the coordinate $\theta = 0$. With respect to that position V is expressed as:

$$V = \frac{1}{2}K_t\theta^2$$

Apply Lagrange's equations, which for purely mechanical systems may be stated as:

$$\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{q}_j}\right) - \frac{\partial(T)}{\partial q_j} + \frac{\partial(V)}{\partial q_j} = Q_j$$

where the q_j are the generalized coordinates. This equation must be applied once for each generalized coordinate in the problem. In this problem, there is only one generalized coordinate, θ. T and V are functions of θ and $\dot{\theta}$. Numbering the terms on the LHS as 1, 2 and 3 in order of appearance, the three terms may be evaluated in a systematic way.

Term 1: $\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{\theta}}\right)=\frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}\left[\frac{1}{2}mL^2\dot{\theta}^2(1+\cos^2\phi)\right]=\frac{d}{dt}\left[mL^2\dot{\theta}(1+\cos^2\phi)\right]=mL^2\ddot{\theta}(1+\cos^2\phi)$

Term 2: $\frac{\partial(T)}{\partial\theta}=0$

Term 3: $\frac{\partial(V)}{\partial\theta}=\frac{\partial}{\partial\theta}\left[\frac{1}{2}K_t\theta^2\right]=K_t\theta$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_θ yields the equation of motion:

$$mL^2\ddot{\theta}(1+\cos^2\phi)+K_t\theta=Q_\theta$$

In this case the amount of virtual work done in a virtual displacement $\delta\theta$ is given by

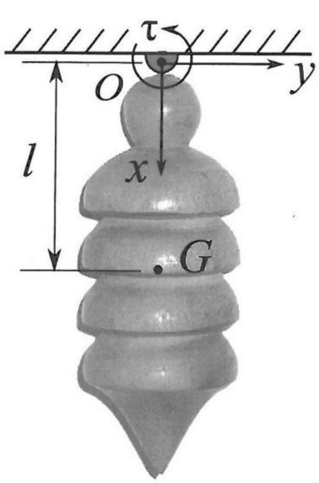
$$Q_\theta\delta\theta=\tau_o(t)\delta\theta$$

$$\Rightarrow Q_\theta=\tau_o(t)$$

$$mL^2\ddot{\theta}(1+\cos^2\phi)+K_t\theta=Q_\theta=\tau_o(t).$$

Problem 3:

A pendulum of mass ‘m’ is allowed to rotate about the z axis passing through point O in the figure. The center of mass is at a distance ‘l’ from O and $I_{zz/G}$ is known. An external time varying torque, $\boldsymbol{\tau}(t)$ is applied to the pendulum at O.



a). Derive the equation(s) of motion using Lagrange equations.

Concept question: Is it appropriate to use the principal axis theorem in this problem? a). yes, b) no.

Solution:

<u>Choose appropriate coordinates:</u> The rotation $\theta(t)\hat{k}$ about the point O is a complete, independent, holonomic set of coordinates for this problem.

Find T and V:

$I_{zz/G}$ is given. We require that a set of principal axes be attached to the rigid body at G. Let the frame attached to the body be called G_{x1y1z1}. Since this is obviously a body with an axis of symmetry in the x_1 direction as drawn in the picture, then both the y_1 and z_1 axes, which are attached to the body and are perpendicular to the x_1 axis are also principal axes. The z_1 axis is coming out of the page at G is parallel to the z axis in the O_{xyz} inertial system fixed at point O. T may be computed using the general formula for the kinetic energy of a rotating and translating rigid body. We may apply the simplifications that are appropriate for planar motion in which the rotation is about a fixed point, which is not the center of mass, as was given by Equation 3 in the appendix to Pset 6. This equation applies to single axis rotation about a fixed axis which is a principal axis.

$$T = \frac{1}{2} m \vec{v}_{G/O} \bullet \vec{v}_{G/O} + \frac{1}{2} I_{ii/G} \omega_i^2$$

In this problem $\vec{v}_{G/O} = l\dot{\theta}\hat{j}_1$

$$I_{ii/G} = I_{zz/G}$$

$$\omega_i = \omega_z$$

$$\therefore T = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} I_{zz/G} \dot{\theta}^2 = \frac{1}{2}\left[m l^2 + I_{zz/G} \right] \dot{\theta}^2$$

The last expression can be recognized as the equivalent to using the parallel axis theorem, where the distance the parallel axis is from G is *l*.

The only source of potential energy in this problem is gravity:

$$V = -\int \vec{\tau} \bullet d\theta \hat{k} = -\int (l \sin\theta \hat{j} \bullet mg\hat{i}) \bullet d\theta \hat{k} = \int_0^\theta mgl \sin\theta d\theta = mgl(1 - \cos\theta)$$

Find the generalized forces: For this problem there is only one generalized force, Q_θ. The virtual work done in a virtual rotation $\delta\theta$ can be figured out by inspection. The only external non-conservative force(or moment) doing work is $\tau(t)$. Therefore
$$\therefore \delta W = \tau(t)\delta\theta = Q_\theta \delta\theta$$
$$\Rightarrow Q_\theta = \tau(t)$$
.

Note that when the external action is a torque, $\tau(t)$, the virtual movement must be a rotation $\delta\theta$.

Apply Lagrange's equations, which for purely mechanical systems may be stated as:

$$\frac{d}{dt}\left(\frac{\partial(T)}{\partial \dot{q}_j} \right) - \frac{\partial(T)}{\partial q_j} + \frac{\partial(V)}{\partial q_j} = Q_j$$
.

where the q_j are the generalized coordinates. This equation must be applied once for each generalized coordinate in the problem. In this problem, there is only one generalized coordinate, θ. T and V are functions of θ and $\dot{\theta}$. Numbering the terms on the LHS as 1, 2 and 3 in order of appearance, the three terms may be evaluated in a systematic way.

Term 1: $\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{\theta}}\right)=\frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}\frac{1}{2}\left[ml^2+I_{zz/G}\right]\dot{\theta}^2=\frac{d}{dt}\left[ml^2+I_{zz/G}\right]\dot{\theta}=\left[ml^2+I_{zz/G}\right]\ddot{\theta}$

Term 2: $\frac{\partial(T)}{\partial\theta}=0$

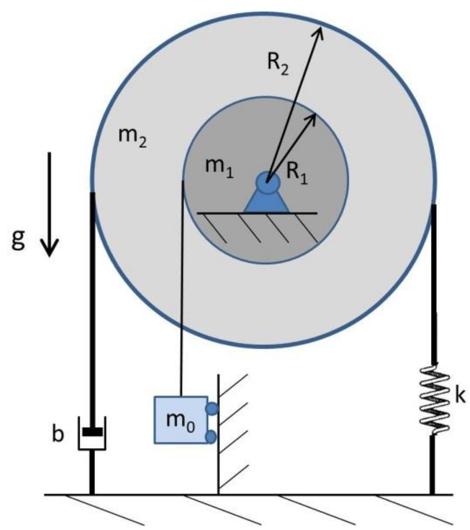
Term 3: $\frac{\partial(V)}{\partial\theta}=\frac{\partial}{\partial\theta}mgl(1-\cos\theta)=mgl\sin\theta$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_θ yields the equation of motion:

$$\left[ml^2+I_{zz/G}\right]\ddot{\theta}+mgl\sin\theta=Q_\theta=\tau(t)$$

Problem 4:

Two uniform cylinders of mass m_1 and m_2 and radius R_1 and R_2 are welded together. This composite object rotates without friction about a fixed point. An inextensible massless string is wrapped without slipping around the larger cylinder. The two ends of the string are connected to a spring of constant k on one end and a dashpot of constant b at the other. The smaller cylinder is connected to a block of mass m_o via an inextensible massless strap, which is wrapped without slipping around the smaller cylinder. The block is constrained to move only vertically.



a) Find the equations of motion using the Lagrange approach.

Concept question: For small values of the dashpot constant, b, if this single degree of freedom system is given an initial displacement from its static equilibrium position, will it exhibit oscillatory motion after release? a)Yes, b)No.

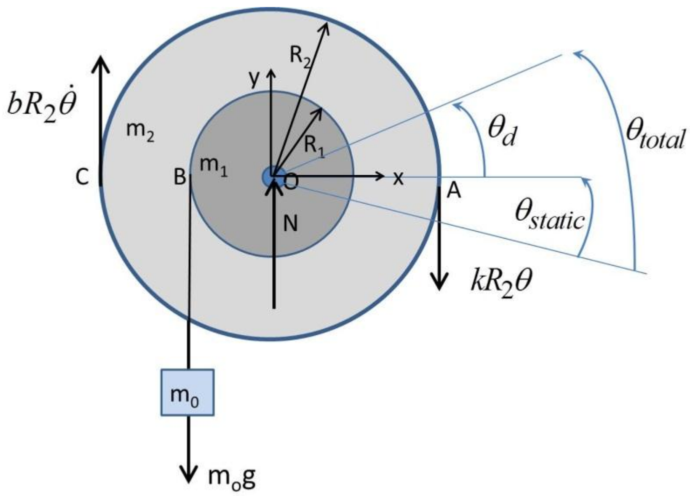
Solution: Specify coordinates: There is one degree of freedom. The coordinate $\theta_d(t)$, which is rotation about the z axis, makes a complete, independent and holonomic set of generalized coordinates for the system.

Find T and V: These were found in Pset 6. They are:

$$T = T_{m_o} + T_{rotor} = \frac{1}{2} m_o \dot{y}^2 + \frac{1}{2} I_{zz/G} \dot{\boldsymbol{\theta}}_d^2 = \frac{1}{2}\left[m_o R_1^2 + m_1 \frac{R_1^2}{2} + m_2 \frac{R_2^2}{2} \right] \dot{\boldsymbol{\theta}}_d^2$$

V= $\frac{1}{2} K R_2^2 \boldsymbol{\theta}_d^2$, where $\boldsymbol{\theta}_d$ is measured from the static equilibrium position. See the figure below from the solution to Pset6.

Find the generalized forces: For this problem there is only one generalized force, Q_θ. The virtual work done in a virtual rotation $\delta\theta_d$ can be figured out by inspection. The only external non-conservative moment doing work during a rotation is due to the dashpot. A positive rotational velocity results in a torque given by $\vec{\tau}_{dashpot} = -bR_2^2 \dot{\theta}_d \hat{k}$. The work done in a positive rotation $\delta\theta$ is given by



$\delta W = \vec{\tau}_{dashpot} \bullet \delta\theta_d \hat{k} = -bR_2^2 \dot{\theta}_d \hat{k} \bullet \delta\theta_d \hat{k} = -bR_2^2 \dot{\theta}_d \delta\theta_d = Q_{\theta_d} \delta\theta_d$. Therefore the generalized force $Q_\theta = -bR_2^2 \dot{\theta}_d$.

Apply Lagrange's equations, which for purely mechanical systems may be stated as:

$$\frac{d}{dt}\left(\frac{\partial(T)}{\partial \dot{q}_j}\right) - \frac{\partial(T)}{\partial q_j} + \frac{\partial(V)}{\partial q_j} = Q_j$$

where the q_j are the generalized coordinates. This equation must be applied once for each generalized coordinate in the problem. In this problem, there is only one generalized coordinate, θ. T and V are functions of θ_d and $\dot{\theta}_d$. Numbering the terms on the LHS as 1, 2 and 3 in order of appearance, the three terms may be evaluated in a systematic way.

Term 1:
$$\frac{d}{dt}\left(\frac{\partial(T)}{\partial \dot{\theta}_d}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}_d}\frac{1}{2}\left[m_o R_1^2 + m_1 \frac{R_1^2}{2} + m_2 \frac{R_2^2}{2} \right] \dot{\theta}_d^2 = \frac{d}{dt}\left[m_o R_1^2 + m_1 \frac{R_1^2}{2} + m_2 \frac{R_2^2}{2} \right] \dot{\theta}_d$$
$$= \left[m_o R_1^2 + m_1 \frac{R_1^2}{2} + m_2 \frac{R_2^2}{2} \right] \ddot{\theta}_d$$

Term 2: $\frac{\partial(T)}{\partial\theta_d} = 0$

Term 3: $\frac{\partial(V)}{\partial\theta_d} = \frac{\partial}{\partial\theta_d}\frac{1}{2}KR_2^2\theta_d^2 = KR_2^2\theta_d$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_θ yields the equation of motion:

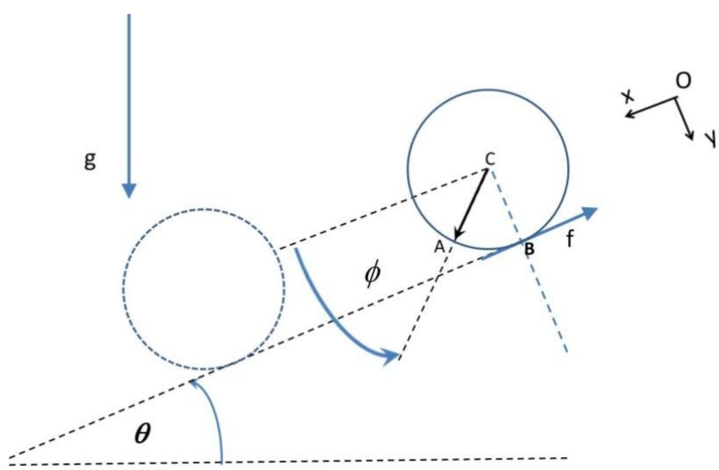
$$\left[m_o R_1^2 + m_1\frac{R_1^2}{2} + m_2\frac{R_2^2}{2}\right]\ddot{\theta}_d + KR_2^2\theta_d = Q_{\theta_d} = -bR_2^2\dot{\theta}_d$$

$$\Rightarrow\left[m_o R_1^2 + m_1\frac{R_1^2}{2} + m_2\frac{R_2^2}{2}\right]\ddot{\theta}_d + bR_2^2\dot{\theta}_d + KR_2^2\theta_d = 0$$

This is the equation of motion of a damped single degree of freedom oscillator. It has no external excitation. If given an initial rotation and released it will exhibit decaying oscillations about the static equilibrium position, which answers the concept question.

Problem 5:

A wheel is released at the top of a hill. It has a mass of 150 kg, a radius of 1.25 m, and a radius of gyration of k_G =0.6 m.



a). Assume that the wheel does not slip as it rolls down the hill. Derive the equation of motion using Lagrange equations.

b). Assume the wheel does slip, and that the friction force between the wheel and the ground is $\vec{f} = -3N\hat{i}$. Derive the equations of motion using Lagrange equations.

Concept question: For which case, a) or b) will friction appear in the equations of motion as a non-conservative generalized force? A. in a) only, B. in b). only, C. in a) and b), D. Never.

Solution: a). Choose the generalized coordinates: With no slip, this is a single degree of freedom system. We can use either the rotation of the wheel or the translation because they are related by $x = R\phi$ and $\dot{x} = R\dot{\phi}$

Compute T and V:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{zz/G} \omega_z^2 = \frac{1}{2} m (R^2 + \kappa_G^2) \dot{\phi}^2$$

We shall use the rotation of the wheel, ϕ, as the single, independent, holonomic coordinate. As the wheel rolls down the hill it decreases in potential energy. Let x=0 be measured from the start at the top of the hill. The vertical distance it drops in a distance x is given by: $h = x \sin\theta$.

$$V = -mgx \sin\theta = -mgR\phi \sin\theta$$

Find the generalized forces: There are no non-conservatives forces which do work in this problem. $Q_\phi = 0$.

Apply Lagrange's Equations to obtain the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial(T)}{\partial \dot{q}_j}\right) - \frac{\partial(T)}{\partial q_j} + \frac{\partial(V)}{\partial q_j} = Q_j \quad (1)$$

Numbering the terms on the LHS as 1, 2 and 3 in order of appearance, the three terms may be evaluated in a systematic way.

Term 1: $$\frac{d}{dt}\left(\frac{\partial(T)}{\partial \dot{\phi}}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{\phi}}\frac{1}{2} m (R^2 + \kappa_G^2)\dot{\phi}^2 = \frac{d}{dt} m (R^2 + \kappa_G^2)\dot{\phi} = m (R^2 + \kappa_G^2)\ddot{\phi} \quad (2)$$

Term 2: $$\frac{\partial(T)}{\partial \theta} = 0 \quad (3)$$

Term 3: $$\frac{\partial(V)}{\partial \phi} = -\frac{\partial}{\partial \phi} mgR\phi \sin\theta = -mgR \sin\theta$$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_θ yields the equation of motion:

$$m (R^2 + \kappa_G^2)\ddot{\phi} - mgR \sin\theta = 0$$
$$\Rightarrow m (R^2 + \kappa_G^2)\ddot{\phi} = mgR \sin\theta$$

This is simply a constant acceleration solution.

b) Part b says to do it again but this time allowing slip.

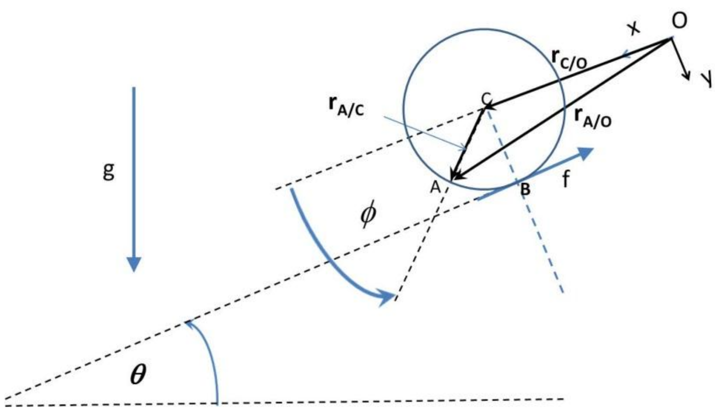
Choose coordinates: With slip, two independent coordinates are required because x and ϕ are not bound by the constraint $x = R\phi$. We choose x and ϕ.

Evaluate T and V:

From part a) there was a suitable expression for T before the no slip constraint was applied in a).

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{zz/G}\omega_z^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\kappa_G^2\dot{\phi}^2$$

$V = -mgx\sin\theta$ where x=0 is taken from the point of release at the top of the hill.



Find the generalized forces: In this problem the only non-conservative force is that caused by friction. In part a) the friction force existed but did not do work on the system and therefore did not appear in the final equation of motion. In this part the friction force does work because of the sliding contact with the ground.

There are two ways to find the generalized forces, the mathematical way and the intuitive way. Both shall be shown here as a learning exercise.

Intuitive approach: In this method we attempt by inspection to determine the small variation, δW_j, of the virtual work done by all of the non-conservative forces, $\mathbf{F_i}$, during a virtual displacement of the generalized force δj.

$$\delta \boldsymbol{W}_j^{nc} = \boldsymbol{Q}_j \delta \boldsymbol{q}_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{q}_j} \right) \delta \boldsymbol{q}_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \delta \boldsymbol{r}_{ij} \qquad (4)$$

$\delta\boldsymbol{r}_{ij}$ is the virtual displacement at the position of the ith non-conservative force, $\mathbf{F_i}$, due to the virtual displacement in the jth generalized coordinate δj. Sometimes this can be done by inspection. In this problem there is only one external force, the friction force, such that $\vec{F}_i = \vec{F}_1 = -f\hat{i}$ **.** During a small virtual displacement δx, the virtual work done by this force is given by $\delta W_x = F_1\ \delta r_{1x} = -f\hat{i}\ \delta_x\hat{i} = -f\delta_x = Q_x\delta_x$. It was easy in this case because it was easy to figure out that $\delta\boldsymbol{r}_{1x} = \delta_x\hat{\boldsymbol{i}}$. We can conclude that $Q_x = -f$. Turning to the second generalized coordinate δ_ϕ, it is slightly more difficult to determine the generalized force Q_ϕ, associated with a small virtual rotation $\delta\phi$. We need to determine $\delta W_\phi = F_1 \bullet \delta r_{1\phi}$. The key is to figure out how $\delta r_{1\phi}$ depends on δ_ϕ. In this case we can deduce pretty quickly that a positive variation δ_ϕ produces a deflection at the point of application of the friction force, given by $\delta r_{1\phi} = -R\delta\phi\hat{i}$.

Therefore $\delta W_\phi = Q_\phi \delta\phi = F_1 \bullet \delta r_{1\phi} = -f\hat{i} \bullet -R\delta\phi\hat{i} = Rf\,\delta\phi$. From which we conclude that $Q_\phi \delta\phi = Rf\,\delta\phi$ and therefore, $Q_\phi = Rf$. Note that since δ_ϕ is a rotation measured in radians, then Q_ϕ must be a torque.

Mathematical approach to finding generalized forces:

The principal equation we need is (4) from the discussion above:

$$\delta \boldsymbol{W}_j^{nc} = \mathbf{Q}_j \delta \boldsymbol{q}_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{q}_j} \right) \delta \boldsymbol{q}_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \delta \boldsymbol{r}_{ij} . \tag{4}$$

The key step is to express the position vector $\mathbf{r_i}$ at each location of the applied non-conservative forces, $\mathbf{F_i}$, in terms of the generalized coordinates so that the partial derivatives called for in the equation above $\frac{\delta r_i}{\delta q_j}$ may be computed.

This is done here for this problem. Consider the point 'A' in the figure above of the wheel at some arbitrary position on the hill. The position of 'A' is determined by coordinates x and ϕ as follows:

$\vec{r}_{A/O} = \vec{r}_{C/O} + \vec{r}_{A/C} = x\hat{i} + R\hat{r}$. The problem with this expression is that not all of the generalized coordinates appear in the expression. ϕ can be brought in by expressing the unit vector, $\hat{r}$, in terms of unit vectors in the O_{xyz} inertial system.

$\hat{r} = \cos\phi\hat{i} + \sin\phi\hat{j}$. This allows the position vector for point 'A' to be expressed in terms of the generalized coordinates, x and ϕ.

$\vec{r}_{A/O} = \vec{r}_{C/O} + \vec{r}_{A/C} = x\hat{i} + R(\cos\phi\hat{i} + \sin\phi\hat{j})$. Now it becomes straightforward to evaluate the partial derivatives needed in equation (4) above.

$$\frac{\delta r_i}{\delta q_j} = \frac{\delta \vec{r}_{A/O}}{\delta x} = \frac{\delta}{\delta x}\left[x\hat{i} + R(\cos\phi\hat{i} + \sin\phi\hat{j}) \right] = \hat{i}$$

$$\frac{\delta \vec{r}_{A/O}}{\delta\phi} = \frac{\delta}{\delta\phi}\left[x\hat{i} + R(\cos\phi\hat{i} + \sin\phi\hat{j}) \right] = R(-\sin\phi\hat{i} + \cos\phi\hat{j})$$

Now it is a straightforward step to evaluate the generalized forces from the expression:

$$\boldsymbol{Q_j} = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial \boldsymbol{r_i}}{\partial \boldsymbol{q_j}} \right) \tag{5}$$

$Q_x = \vec{F}_1 \bullet \left(\dfrac{\partial r_1}{\partial q_x} \right) = -f\hat{i} \bullet \hat{i} = -f$ and

$Q_\phi = \vec{F}_1 \bullet \left(\dfrac{\partial r_1}{\partial q_\phi} \right) = -f\hat{i} \bullet R(+\cos\phi\,\hat{j} - \sin\phi\,\hat{i}) = Rf\sin\phi\,|_{\phi=\pi/2} = Rf$

Note that it is only at the last step that the specific value of $\phi = \pi/2$ is used to specify the location of the point of application of the external friction force. One must compute the partial derivatives before substituting in specific values of the generalized coordinates, which correspond to the specific locations of the applied external non-conservative forces.

<u>Find the equations of motion using Lagrange's equations:</u> This must be done twice in this problem, once for each generalized coordinate. Here we do the equation for the coordinate x first.

Generalized coordinate x:

$$\frac{d}{dt}\left(\frac{\partial (T)}{\partial \dot{q}_j} \right) - \frac{\partial (T)}{\partial q_j} + \frac{\partial (V)}{\partial q_j} = Q_j .$$

Numbering and evaluating the terms on the LHS as 1, 2 and 3 in order of appearance, leads to:.

Term 1: $\dfrac{d}{dt}\left(\dfrac{\partial (T)}{\partial \dot{\phi}} \right) = \dfrac{d}{dt}\dfrac{\partial}{\partial \dot{\phi}}\left[\dfrac{1}{2} m\dot{x}^2 + \dfrac{1}{2} m\kappa_G^2 \dot{\phi}^2 \right] = \dfrac{d}{dt}\left[m\kappa_G^2 \dot{\phi} \right] = \left[m\kappa_G^2 \ddot{\phi} \right]$

Term 2: $\dfrac{\partial (T)}{\partial \phi} = 0$

Term 3: $\dfrac{\partial (V)}{\partial \phi} = \dfrac{\partial}{\partial \phi} - mgx\sin\theta = 0$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_θ yields the equation of motion:

$m\kappa_G^2 \ddot{\phi} = Rf = 1.25m(3N) = 3.75N - m$

Generalized coordinate ϕ:

Term 1: $\dfrac{d}{dt}\left(\dfrac{\partial (T)}{\partial \dot{x}} \right) = \dfrac{d}{dt}\dfrac{\partial}{\partial \dot{x}}\left[\dfrac{1}{2} m\dot{x}^2 \right] = \dfrac{d}{dt}\left[m\dot{x} \right] = m\ddot{x}$

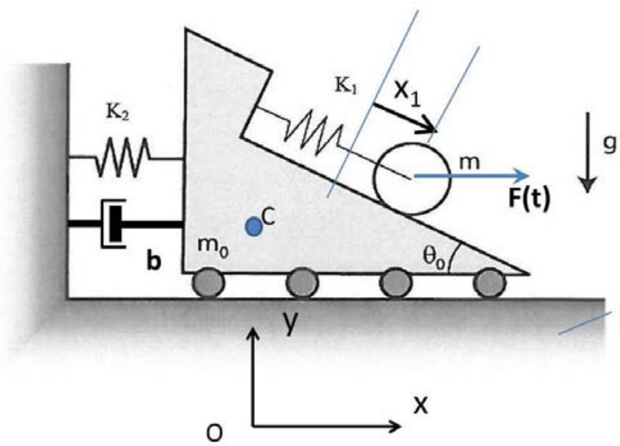
Term 2: $\frac{\partial(T)}{\partial x} = 0$

Term 3: $\frac{\partial(V)}{\partial x} = \frac{\partial}{\partial x}\left(-mgx\sin\theta\right) = -mg\sin\theta$

Summing terms 1, 2 and 3 from the LHS and equating it to the generalized force Q_x yields the equation of motion:

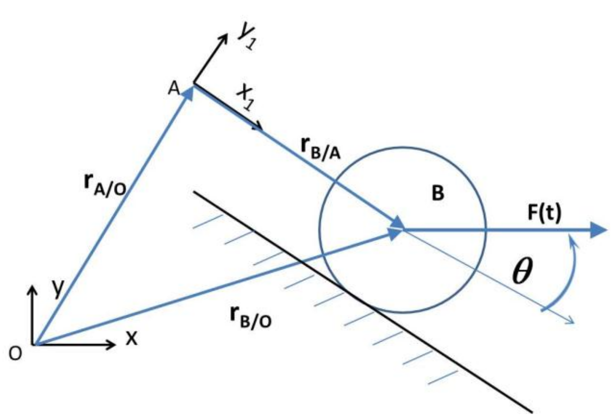
$m\ddot{x} - mg\sin\theta = -f$, which in more standard form is
$m\ddot{x} = mg\sin\theta - f = mg\sin\theta - 3N$

Problem 6: The cart shown in the figure has mass m_o. It has an inclined surface as shown. A uniform disk of mass m, and radius R, rolls without slip on the inclined surface. The disk is restrained by a spring, K_1, attached at one end to the cart. The other end of the spring attaches to an axel passing through the center of the disk. The cart is also attached to a wall by a spring of constant, K_2, and a dashpot with constant 'b'. An horizontal external force, F(t), is applied at the center of mass of the disk as shown.



a). Derive the equations of motion for the system using Lagrange's equations, including the non-conservative forces. Recall in PSet6 Tand V were found.

Concept question: If the coordinate 'x' represents the horizontal motion of the cart, what is the generalized force associated with F(t) due to a virtual displacement δx ? a). $F(t)\cos\theta$, b). F(t), c). $F(t)\delta x$, d). 0.



Solution: In Pset6 the equations of motion for this system were found using Lagrange's equations, for the case that there were no external non-conservative generalized forces. What was found for that case is still valid here for the left hand side of the Lagrange equation result. All that remains to be done is to evaluate the generalized forces which do exist in this problem. First, the LHS from Pset6. Recall there are two degrees of freedom and two generalized coordinates, x and x_1. The two equations of motion which were found previously are:

x equation: $(m_o + m)\ddot{x} + m\ddot{x}_1 \cos\theta_o + k_2 x = 0$,

x_1 equation: $\frac{3}{2}m\ddot{x}_1 + m\ddot{x}\cos\theta_o + k_1 x_1 = 0$,

Finding the generalized forces:

There are two generalized forces to be found, Q_x and Q_{x1}, associated with the two equations above. It is reasonably simple to determine them by deduction. When using this method proceed with one virtual displacement of a generalized coordinate at a time and require that all other virtual displacements be held at zero.

In this case, find first the generalized force Q_x which is associated with the virtual work done in a small virtual displacement δx. Assume for the moment that the other possible virtual displacement $\delta x_1 = 0$.

$$\delta W_x = Q_x \delta x = (-b\dot{x}\hat{i} + F(t)\hat{i}) \bullet \delta x\hat{i} = (-b\dot{x} + F(t))\delta x$$
$$\Rightarrow Q_x = -b\dot{x} + F(t)$$

Therefore the first equation of motion becomes:

$$(m_o + m)\ddot{x} + m\ddot{x}_1 \cos\theta_o + k_2 x = Q_x = -b\dot{x} + F(t)$$

which upon rearrangement:

$$(m_o + m)\ddot{x} + b\dot{x} + m\ddot{x}_1 \cos\theta_o + k_2 x = F(t)$$

To determine the second generalize force associated with coordinate x_1 requires that we express the force the external force F(t) in terms of the unit vectors associated with the moving coordinate system A_{x1y1z1}, which is attached to the cart.

$$F(t)\hat{i} = F(t)(\cos\theta_o \hat{i}_1 + \sin\theta_o \hat{j}_1).$$

This makes it possible to compute the virtual work done in a virtual displacement of the coordinate x_1, as follows:

$$\delta W_{x_1} = Q_{x_1}\delta x_1 = (F(t)(\cos\theta_o \hat{i}_1 + \sin\theta_o \hat{j}_1)\ \delta x_1 \hat{i}_1 \bullet = F(t)\cos\theta_o \delta x_1$$
$$\Rightarrow Q_{x_1} = F(t)\cos\theta_o$$

With this we can complete the second equation of motion:

$$\frac{3}{2}m\ddot{x}_1 + m\ddot{x}\cos\theta_o + k_1 x_1 = Q_{x_1} = F(t)\cos\theta_o$$

It is also possible to find the generalized forces by the more rigorous mathematical method expressed in the following equation:

$$\boldsymbol{Q_j} = \sum_{\boldsymbol{i}=1}^{\boldsymbol{N}} \mathbf{F}_{\boldsymbol{i}} \bullet \left(\frac{\partial \boldsymbol{r_i}}{\partial \boldsymbol{q_j}} \right)$$

In this problem there are two points of application of the external non-conservative forces, the dashpot and the external force,3 F(t). Let them be represented by:

$$F_1 = -b\dot{x}\hat{i}$$
$$F_2 = F(t)\hat{i}$$

It is necessary to find the position vectors associated with each force:

$\vec{r}_1 = x\hat{i} + y\hat{j}$, where y is a constant.

$\frac{\delta \vec{r}_1}{\delta x} = \hat{i}$, and $\frac{\delta \vec{r}_1}{\delta x_1} = 0$

Evaluating $Q_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial r_i}{\partial q_j} \right)$ leads to:

$$\mathrm{Q}_{1,x} = (-b\dot{x}\hat{i})\bullet\hat{i} = -b\dot{x}$$
$$Q_{1,x_1} = 0$$

$$\vec{r}_2 = \vec{r}_{A/O} + \vec{r}_{B/A} = x\hat{i} + y\hat{j} + x_1\hat{i}_1$$

$\frac{\delta \vec{r}_2}{\delta x} = \hat{i}$, and $\frac{\delta \vec{r}_2}{\delta x_1} = \hat{i}_1 = \cos\theta_o\hat{i} + \sin\theta_o\hat{j}$

Evaluating $Q_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial r_i}{\partial q_j} \right)$ leads to

$$\mathrm{Q}_{2,x} = F\hat{i}\bullet\hat{i} = F$$
$$Q_{2,x_1} = F\hat{i}\bullet(\cos\theta_o\hat{i} + \sin\theta_o\hat{j}) = F\cos\theta_o$$

As found by the intuitive approach:

$Q_x = -b\dot{x} + F(t)$ and $Q_{x_1} = F(t)\cos\theta_o$.