

2.003 Engineering Dynamics

Problem Set 6 with solution

Problem 1: A slender uniform rod of mass m_2 is attached to a cart of mass m_1 at a frictionless pivot located at point „A“. The cart is connected to a fixed wall by a spring and a damper. The

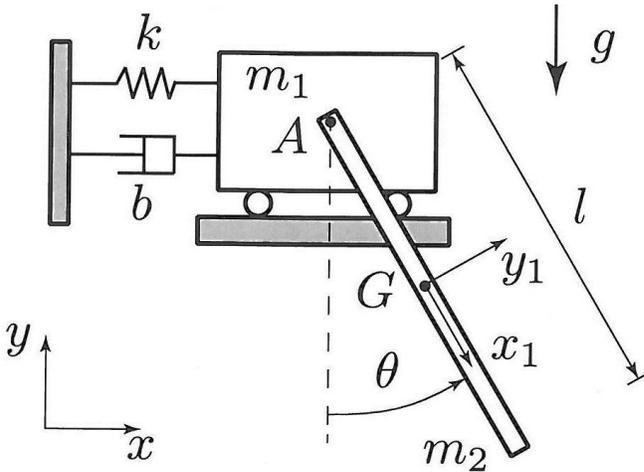


Figure 1. Cart with a slender rod

cart rolls without friction in the horizontal direction. The position of the cart in the inertial frame O_{xyz} is given by $x\hat{i}$.

a). Assuming that the motion of the cart and slender rod is the result of initial conditions only, find expressions for T and V , the kinetic and potential energy of the system in terms of x , \dot{x} , θ , and $\dot{\theta}$.

Concept question: The kinetic energy of the rod may be expressed by the

equation $T_{rod} = \frac{1}{2} I_{zz/A} \dot{\theta}^2$: a). True, b).

False. Answer: False—the rod also has velocity terms involving \dot{x} , the velocity of the cart.

Solution: This is a planar motion problem. However, although the axis of rotation is perpendicular to the plane of the motion, the axis of rotation is not at a fixed point. Therefore to

find T it is best to use the form $T = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \vec{I}_{/G} \vec{\omega}$ (Eq. (1) in appendix)

Since there are two rigid bodies, it is necessary to apply this equation twice: first for the cart and second for the rod rotating about the axis at „A“ on the cart.

$$T_{cart} = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \vec{I}_{/G} \vec{\omega} = \frac{1}{2} m \dot{x}^2$$

$$T_{rod} = \frac{1}{2} m_2 \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_{I/G} \text{ where } \vec{v}_{G/O} = \vec{v}_{A/O} + \vec{v}_{G/A} = \dot{x} \hat{i} + \dot{\theta} \hat{k} \times \frac{L}{2} \hat{i}_1 = \dot{x} \hat{i} + \frac{L}{2} \dot{\theta} \hat{j}_1$$

$$\frac{1}{2} m_2 \vec{v}_{G/O} \cdot \vec{v}_{G/O} = \frac{1}{2} m_2 \left[\dot{x}^2 + \frac{L^2}{4} \dot{\theta}^2 + L \dot{x} \dot{\theta} \cos \theta \right]$$

$$\frac{1}{2} \vec{\omega} \cdot \vec{H}_{I/G} = \frac{1}{2} I_{zz/G} \dot{\theta}^2 = m_2 \frac{L^2}{24} \dot{\theta}^2$$

$$T_{rod} = \frac{1}{2} m_2 \left[\dot{x}^2 + \frac{L^2}{3} \dot{\theta}^2 + L \dot{x} \dot{\theta} \cos \theta \right]$$

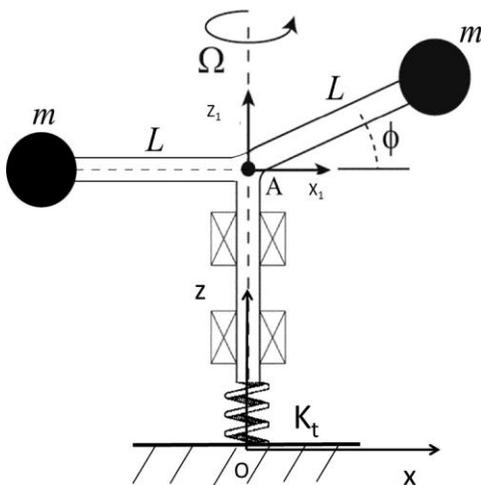
$$T = T_{cart} + T_{rod} = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 \left[\frac{L^2}{3} \dot{\theta}^2 + L \dot{x} \dot{\theta} \cos \theta \right]$$

$$V = \frac{1}{2} kx^2 + m_2 g \frac{L}{2} (1 - \cos \theta) \text{ where } x=0 \text{ is at the unstretched spring position, which is also the}$$

static equilibrium position for the cart. The static equilibrium position for the rod is when it is hanging straight down. The coordinate θ is chosen so that it is zero at static equilibrium. In this way the potential function, V , for this problem is zero at static equilibrium.

Problem 2:

Two identical masses are attached to the end of massless rigid arms as shown in the figure. The vertical portion of the rod is held in place by bearings that prevent vertical motion, but allow the shaft to rotate without friction. A torsion spring with spring constant K_t resists rotation of the vertical shaft. The shaft rotates with a time varying angular velocity Ω with respect to the O_{xyz} inertial frame. The arms are of length L . The frame $A_{x_1y_1z_1}$ rotates with the arms and attached masses. Note that the angle ϕ is fixed.



a) Find T and V , the kinetic and potential energy for this system.

Concept question: Is it possible to find the equation of motion of this system by requiring that:

$\frac{d}{dt} [T + V] = 0$. a). Yes, b). No. Answer: Yes, this is a useful approach for single dof problems with no non-conservative forces.

Solution: This rigid body rotates about a fixed axis passing through „A“. However, it is not a planar motion problem. We may either use the full 3D formulation for the computation of T for

a rigid body as given by $T = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \mathbf{I}_{G/O} \vec{\omega}$ or because it rotates about a fixed axis at

„A“ we may use the form $T = \frac{1}{2} \vec{\omega} \cdot \mathbf{I}_{/A} \vec{\omega} = \frac{1}{2} \vec{\omega} \cdot [\mathbf{I}_{/A}] \{\omega\}$, which is easier in this case because in

Pset 5 we found $\vec{H}_{/A} = [\mathbf{I}_{/A}] \{\omega\}$, where

$$[\mathbf{I}_{/A}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} mL^2 \sin^2 \phi & 0 & -mL^2 \sin \phi \cos \phi \\ 0 & 2mL^2 & \\ -mL^2 \sin \phi \cos \phi & & mL^2(1 + \cos^2 \phi) \end{bmatrix} \text{ and } \omega = \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix}.$$

$$\{\vec{H}_{/A}\} = \begin{Bmatrix} H_x \hat{i}_1 \\ H_y \hat{j}_1 \\ H_z \hat{k}_1 \end{Bmatrix} = [\mathbf{I}_{/A}] \{\omega\} = [\mathbf{I}_{/A}] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [\mathbf{I}_{/A}] \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} = \begin{Bmatrix} -mL^2 \Omega \sin \phi \cos \phi \hat{i}_1 \\ 0 \\ mL^2 \Omega (1 + \cos^2 \phi) \hat{k}_1 \end{Bmatrix}$$

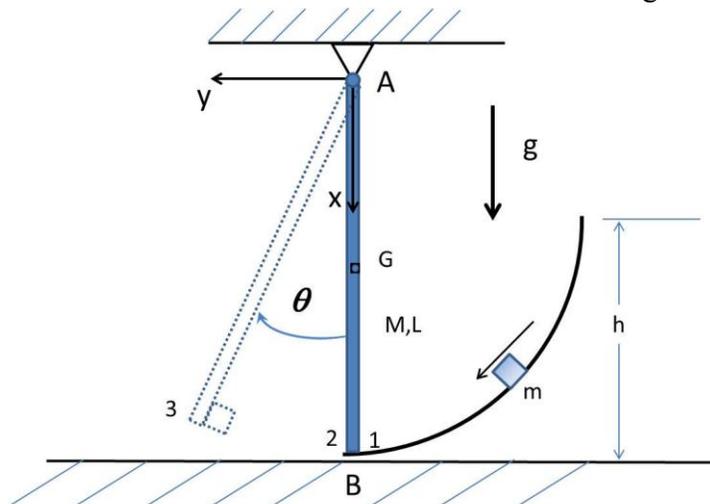
It only remains to compute T from the expression

$$T = \frac{1}{2} \vec{\omega} \cdot \mathbf{I}_{/A} \vec{\omega} = \frac{1}{2} \begin{bmatrix} 0 & 0 & \Omega \hat{k}_1 \end{bmatrix} \cdot \begin{Bmatrix} -mL^2 \Omega \sin \phi \cos \phi \hat{i}_1 \\ 0 \\ mL^2 \Omega (1 + \cos^2 \phi) \hat{k}_1 \end{Bmatrix} = \frac{1}{2} mL^2 \Omega^2 (1 + \cos^2 \phi)$$

The potential energy comes from the torsional spring. The zero spring torque position of the rotor is selected as the reference position for $V=0$. When the system is in static equilibrium, the coordinate $\theta=0$. Therefore,

$$V = \frac{1}{2} K_t \theta^2$$

Problem 3: A particle of mass m slides down a frictionless surface. It then collides with and sticks to a uniform vertical rod of mass M and length L . Following the collision, the rod pivots about the point O . Point G is the mass center of the rod.



a). Find the kinetic energy, T , and the potential energy, V , of the system after the collision as a function of θ and $\dot{\theta}$.

Concept question: Angular momentum with respect to point A and $(T+V)$ are both constant after the

collision. A). True, B)False. Answer: False, T+V= constant, but angular momentum is not.

Solution:

To compute the angular velocity prior to the collision can be obtain by conservation of energy. Let T_o and V_o be the potential and kinetic energy at the time of release at the top of the slope. At this time let $V_o=0$. $T_o=0$ because it is at zero speed. There is no friction so total energy is conserved. Let points 1 and 2 in the figure above correspond to $\theta = 0$ just before(1) and just after(2) the collision. Let state 3 be at any allowable position, θ , after the collision. Therefore,

$$T_o + V_o = 0 = T_1 + V_1 \Rightarrow T_1 = -V_1$$

Where

$$T_1 = \frac{1}{2} m v_{1/A}^2 = -V_1 = mgh$$

$$\Rightarrow v_{1/A} = \sqrt{2gh} = v_{1/O}, \text{ because A is a non-moving inertial point.}$$

The angular momentum of the mass, m, at point 1 is simply

$$H_{1/A} = \vec{r}_{B/A} \times \vec{P}_{1/O} = L\hat{r} \times m\vec{v}_{1/O} = L\hat{r} \times m\sqrt{2gh}\hat{\theta} = -mL\sqrt{2gh}\hat{k}$$

Because angular momentum is conserved during the collision:

$$\vec{H}_{1/A} = \vec{H}_{2/A} = I_{zz/A} \dot{\theta}_2 \hat{k}$$

$$\text{Where after the collision } I_{zz/A} = mL^2 + M \frac{L^2}{3} = (m + M / 3)L^2$$

$$\vec{H}_{1/A} = mL\sqrt{2gh}\hat{k} = (m + M / 3)L^2 \dot{\theta}_2 \hat{k} = \vec{H}_{2/A}$$

$\therefore \dot{\theta}_2$ at point 2 may be solved for, yielding the expression

$$\dot{\theta}_2 = \frac{m}{m + M / 3} \frac{\sqrt{2gh}}{L}$$

The last step is to realize that after the collision total energy is conserved, as may be expressed by the equation

$$T_2 + V_2 = T_3 + V_3$$

For this part of the problem, let the potential energy reference position be such that when $\theta = 0$, then $V_2 = 0$. Substituting into the equation above, yields:

$$T_2 = \frac{1}{2} \vec{\omega} \mathbf{I}_{2/A} \vec{\omega} = \frac{1}{2} \dot{\theta}_2 \hat{k} \mathbf{I}_{2/A} \dot{\theta}_2 \hat{k} = \frac{1}{2} [(m + M/3)L^2 \dot{\theta}_2^2] = \frac{1}{2} \left[\frac{m^2 2gh}{m + M/3} \right] = T_3 + V_3$$

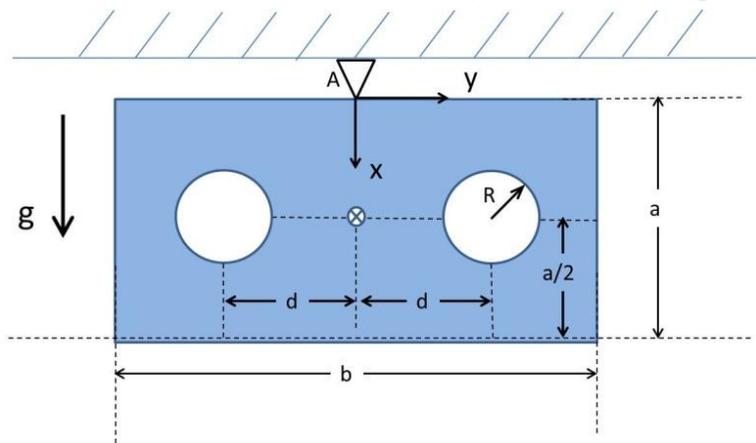
$$T_3 = \frac{1}{2} [(m + M/3)L^2 \dot{\theta}^2]$$

$$V_3 = (mgL + ML/2)(1 - \cos \theta)$$

The maximum angle of θ , could, for example be solved for by requiring $V_3 = T_2$.

Problem 4:

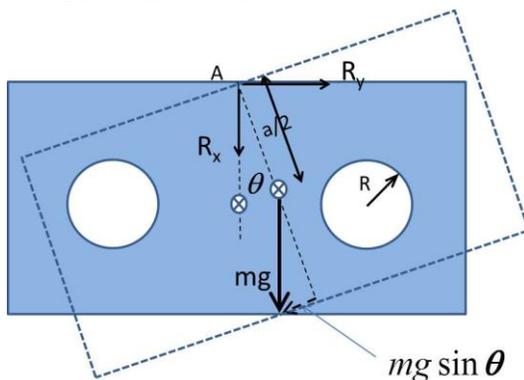
A pendulum consists of a rectangular plate (of thickness t) made of a material of density ρ , with two identical circular holes (of radius R). The pivot is at A.



- a). Find expressions for T and V, the kinetic and potential energies of the system in terms of the angle of rotation about point A.
- b). This is a planar motion problem. How many degrees of freedom does this body have. What are the constraints on this rigid body.

Concept question: According to a strict definition of „translation“: “All points on a rigid body must travel in parallel paths”, does this rigid body exhibit rigid body translation?: a). Yes, b). No, Answer: No, By strict definition, this object exhibits pure rotation about A. All points move in circular paths.

Solution: a). It is always correct for rigid bodies to use the general 3D formula for kinetic energy, T, given by



$T = \frac{1}{2} m \vec{v}_{G/O} \mathbf{i}_{G/O} + \frac{1}{2} \vec{\omega} \mathbf{I}_{G} \vec{\omega}$. However, since this is planar motion about a fixed axis passing through „A“ and this axis is a principal axis, then the following simpler expression may be used.

$$T = \frac{1}{2} \dot{\theta}^T [I_{/A}] \{\dot{\theta}\} = \frac{1}{2} I_{zz/A} \dot{\omega}_z^2 \text{ where } \omega_z = \dot{\theta}.$$

$$I_{zz/A, \text{with holes}} = I_{zz/C, \text{with holes}} + M_{\text{with holes}} \frac{a^2}{4} \text{ (by parallel axis theorem)}$$

$$= \rho abt \frac{(a^2 + b^2)}{12} - 2\rho\pi R^2 t \frac{R^2}{2} - 2\rho\pi R^2 t d^2 + \rho[abt - 2\pi R^2 t] \frac{a^2}{4}$$

$$I_{zz/A, \text{with holes}} = \rho abt \left[\frac{b^2}{12} + \frac{a^2}{3} \right] - 2\rho\pi R^2 t \left[\frac{R^2}{2} + d^2 + \frac{a^2}{4} \right]$$

Let the reference position for V be $\theta = 0$. Then V for the system is given by:

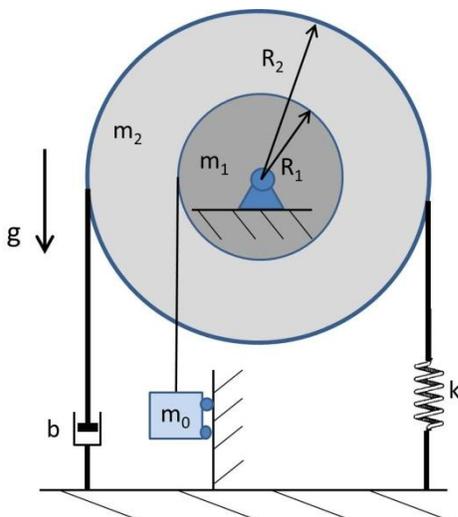
$$V = (M - 2m)g \frac{a}{2} (1 - \cos \theta)$$

$$V = \rho t (ab - 2\pi R^2) g \frac{a}{2} (1 - \cos \theta)$$

b). This is planar motion about a fixed axis. Only one coordinate is required to completely describe the motion. That coordinate is θ . In planar motion each rigid body has at most 3 degrees of freedom, two in translation and one in rotation. In this problem because the body is not allowed to translate in x or y at the pivot point „A“ it has two translational constraints.

Problem 5:

Two uniform cylinders of mass m_1 and m_2 and radius R_1 and R_2 are welded together. This composite object rotates without friction about a fixed point. An inextensible massless string is wrapped without slipping around the larger cylinder. The two ends of the string are connected to the ground via, respectively, a spring of constant k and a dashpot of constant b . The smaller cylinder is connected to a block of mass m_0 via an inextensible massless strap wrapped without slipping around the smaller cylinder. The block is constrained to move only vertically.



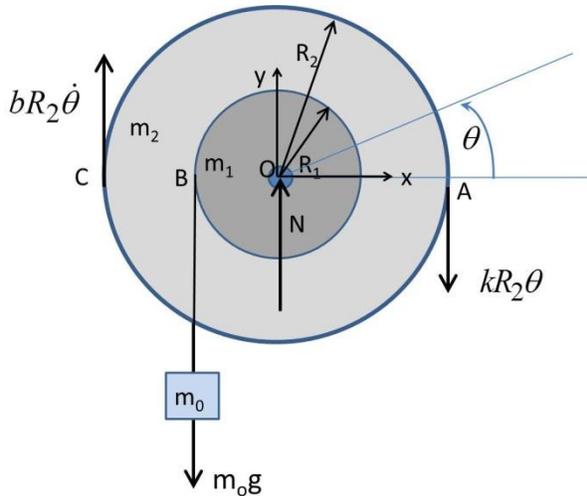
a) Find expressions for T and V the kinetic and potential energy of the system.

Concept question: For small values of the dashpot constant, b , if this single degree of freedom system is

given an initial displacement from its static equilibrium position, will it exhibit oscillatory motion after release? a)Yes, b)No. Answer: Yes, it will oscillate about its static equilibrium position.

Solution:

This system consists of two rigid body masses, and yet requires only one independent coordinate to completely describe the motion of the system. The coordinate to be used in this solution is the rotation angle, θ , of the wheel about the fixed axis at the center of the wheel.



If the wheel rotates through an angle θ then the small mass moves a distance $y = -R_1\theta$. The strap connecting the mass to the wheel is a constraint. This is also a planar motion problem. The figure at left is the free body diagram from PSet5. It will be helpful in working out the relationships we need to find the kinetic and potential energy of the system.

Because of the spring, this system will oscillate about a static equilibrium position. At rest the

spring must exert a torque which exactly balances the torque caused by the hanging weight. By referring to the free body diagram above, the torques about the axis of rotation at „O“ may be identified and required to sum to zero. Expressed as an equation:

$$\sum_{static} \tau_{/O} = 0 = -kR_2\theta_{static} + m_0gR_1 \Rightarrow \theta_{static} = \frac{m_0gR_1}{kR_2}. \tag{1}$$

It is usually advantageous to write the equation of motion such that the motion variable is zero at equilibrium. In this case, let $\theta_d = 0$ when the system is at rest in static equilibrium, where the subscript „d“ refers to the dynamic rotation. Therefore the total rotation from the zero spring force position is given by $\theta_{total} = \theta_{static} + \theta_d$, by taking time derivatives we can easily show that

$$\dot{\theta}_{total} = \dot{\theta}_d \text{ and } \ddot{\theta}_{total} = \ddot{\theta}_d. \tag{2}$$

Now find T and V with θ_d as the dynamic motion coordinate.

In general it is much more straightforward to work out the kinetic energy expression than the potential. It is done first here.

$$T = T_{m_0} + T_{rotor} = \frac{1}{2}m_0\dot{y}^2 + \frac{1}{2}I_{zz/G}\dot{\theta}_d^2 = \frac{1}{2}\left[m_0R_1^2 + m_1\frac{R_1^2}{2} + m_2\frac{R_2^2}{2}\right]\dot{\theta}_d^2 \tag{3}$$

The evaluation of the potential energy requires that the analyst make choices that will effect the final form of the equation of motion. One of the choices is what to use as the reference level for the potential energy function. In this case the reference level will be associated with the choice of the measuring the angle of rotation with respect to the static equilibrium position of the system.

The potential energy of the system is defined with the following equation, which states that the potential energy of the system is given by the negative of the work done by the external conservative forces in the system. For mechanical systems, the only conservative forces are those from gravity and from springs.

$$V = \sum \left[-\int T d\theta_d - \int F dr \right] \quad (4)$$

In this case the work is most directly expressed in terms of torque. Therefore,

$$V = \sum \left[-\int T d\theta_d \right] = \int_0^{\theta_d} \left[-m_o g R_1 \hat{k} + K R_2^2 (\theta_d + \theta_{static}) \right] d\theta_d$$

$$\Rightarrow V = \int_0^{\theta_d} \left[(-m_o g R_1 \hat{k} + K R_2^2 \theta_{static}) + K R_2^2 \theta_d \right] d\theta_d = \int_0^{\theta_d} \left[K R_2^2 \theta_d \right] d\theta_d \quad (5)$$

because from static equilibrium as expressed in equation (1)

$$(-m_o g R_1 \hat{k} + K R_2^2 \theta_{static}) = 0$$

$$\text{and therefore } V = \frac{1}{2} K R_2^2 \theta_d^2$$

Careful choice of the reference position has yielded an exceptionally simple expression for the potential energy V. In fact it does not even involve gravity. An experienced dynamicist would know to expect that gravity could be eliminated from the EOM just by finding the static equilibrium condition expressed in equation (1).

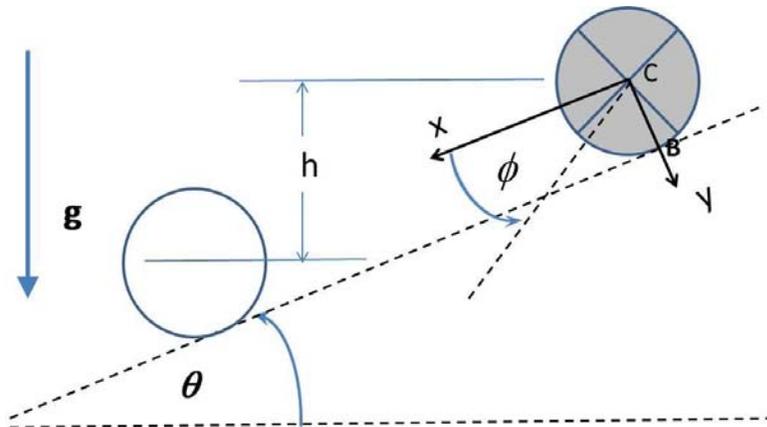
Here is the general rule: Any time the gravity term in the equation of static equilibrium does NOT change as the motion coordinate is varied, it is possible to select a reference position for that coordinate so that a gravity term will not appear in the final equation of motion. In this problem the motion variable (coordinate) is the angle of rotation. When summing external torques in equation (1) the torque caused by gravity was simply $\tau_{/O} = m_o g R_1 \hat{k}$, which is not a function of the angle of rotation of the rotor. A counter example would be a simple pendulum. There the torque caused by gravity changes with the angle the pendulum makes with the vertical.

For this problem the bottom line answer is:

$$T = \frac{1}{2} \left[m_o R_1^2 + m_1 \frac{R_1^2}{2} + m_2 \frac{R_2^2}{2} \right] \dot{\theta}_d^2$$

$V = \frac{1}{2} K R_2^2 \theta_d^2$, where θ_d is measured from the static equilibrium position.

Problem 6:



A wheel is released at the top of a hill. It has a mass of 150 kg, a radius of 1.25 m, and a radius of gyration of $k_G = 0.6$ m.

a). After release from the top of the hill the wheel rolls without slip down the hill. Find the kinetic energy, T , after the center of mass of the wheel has descended a vertical height „ h “.

b). Compute the ratio of the translational kinetic energy to the total kinetic energy of the system.

Concept question a) If the wheel slips during its passage down the hill, is it correct to model it as a planar motion problem. A). Yes, b). No, Answer: Yes, its translational motion is confined to the x-y plane and the axis of rotation is a principal axis which is perpendicular to the x-y plane.

Solution:

Without slip this rigid body has one degree of freedom and therefore requires only one independent coordinate to completely describe the motion. The no slip condition provides a relationship between translation and rotation of the wheel: $x = R\phi$ and $\dot{x} = R\dot{\phi}$.

In general the kinetic energy of a rigid body can be written as:

$$T = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_{I/G}, \tag{1}$$

which is valid for 3D motion of any rigid body.

If the body rotates about a fixed axis passing through a point „A“ then one may substitute the following formula:

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{H}_{I/A} = \frac{1}{2} \vec{\omega} \cdot [I_{I/A}] \{\omega\} \tag{2}$$

For planar motion problems when the axis of rotation, passing through „A“ is a principal axis the formula becomes simpler still. This is a planar motion problem and the wheel has an axis of symmetry about which it rotates. That axis is a principal axis. T may be written using the simple planar motion equation for T about a fixed axis.

$$T = \frac{1}{2} \bar{\omega} \mathbf{I}_{i/A} \{\omega\} = \frac{1}{2} I_{i/A} \omega_i^2 \quad \text{where } i \text{ is the axis of rotation.} \quad (3)$$

For planar motion problems in which the body rotates about a principal axis passing through the center of mass, G, then it is also appropriate to use the form $T = \frac{1}{2} m \bar{v}_{G/O}^2 + \frac{1}{2} I_{i/G} \omega_i^2$ (4)

A wheel rolling without slip may use either equation (3) or (4). In this case $\{\omega\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \omega_z \end{Bmatrix}$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{zz/G} \omega_z^2 = \frac{1}{2} m (R\dot{\theta})^2 + \frac{1}{2} m \kappa_G^2 \dot{\theta}^2 \quad (4)$$

Let state 1 be at the release point and state 2 be after descending a vertical distance „h“, the following is an expression of conservation of total system energy:

$$T_1 + V_1 = T_2 + V_2, \quad \text{where } V_1 = T_1 = 0 \Rightarrow T_2 = -V_2$$

Let s be a vertical coordinate, positive downward. s=0 at the top of the hill. The potential energy after descending a distance „h“ is therefore,

$$V_2 = -\int_0^h mg \hat{s} ds = -mg \int_0^h ds = -mgh$$

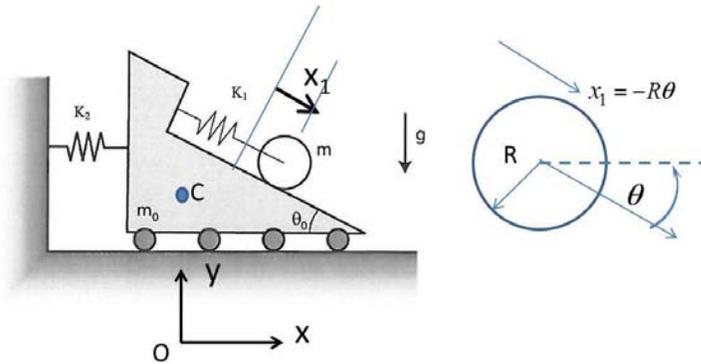
$$T_2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{zz/G} \omega_z^2 = \frac{1}{2} m (R^2 + \kappa_G^2) \dot{\theta}^2 = -V_2 = mgh$$

$$\Rightarrow \dot{\theta}^2 = \frac{2gh}{(R^2 + \kappa_G^2)}$$

Part (b). The first term in equation (4) above is the kinetic energy associated with translation and the second term is the kinetic energy associated with rotation about the center of mass. The ratio of the translational kinetic to the total kinetic energy is given by:

$$\frac{T_{trans}}{T} = \frac{\frac{1}{2} m (R\dot{\theta})^2}{\frac{1}{2} m (R\dot{\theta})^2 + \frac{1}{2} m \kappa_G^2 \dot{\theta}^2} = \frac{R^2}{R^2 + \kappa_G^2}$$

Problem 7: The cart shown in the figure has mass m_0 . It has an inclined surface as shown. A uniform disk of mass m , and radius R , rolls without slip on the inclined surface. The disk is restrained by a spring, K_1 , attached at one end to the cart. The other end of the spring attaches to an axel passing through the center of the disk. The cart is restrained by a second spring, K_2 , which is attached to a non-moving wall.

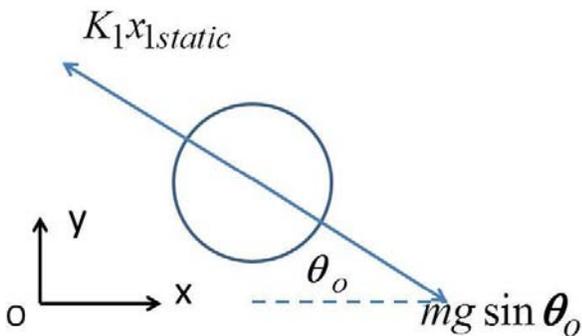


- Find expressions for the kinetic energy, T , and potential energy, V , for the system.
- Use Lagrange equations to find the equations of motion of the system.

Concept question: How many independent coordinates are required to account for the kinetic and potential energy in the system: a). 1, b). 2, c). 3. Answer: b). This system has two degrees of freedom and requires two independent coordinates.

Solution:

This is a two degree of freedom system. Two independent coordinates are required. Let x be the position of the cart in the horizontal direction relative to the inertial frame. Let $x=0$ be the static equilibrium position of the cart. In this case this coincides to the zero spring force position as well. The potential energy stored in the spring is simply $V_{k_2} = \frac{1}{2} k_2 x^2$. The second coordinate is x_1 , the position of the roller relative to the cart. Let $x_1=0$ be the static equilibrium position of the roller. A free body diagram of the roller at static equilibrium is shown in the first figure below



for forces in the direction of coordinate x_1 . Below that is a free body diagram for the dynamic case when x_1 is the movement of the wheel about the static position.

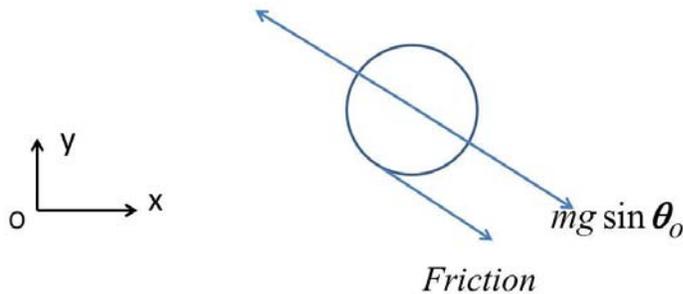
- Begin by computing the potential energy function. Note that in the free body diagram for the dynamic case, the static force in the spring exactly cancels the gravitational force on the object. The only term left in the potential energy

term associated with the motion of the roller, V_{wheel} , is that due to the stretch of the spring from the static equilibrium position. The potential energy function V is independent of gravity. This will simplify the final equation of motion.

$$V_{wheel} = -\int_0^{x_1} \vec{F} \cdot d\vec{x}_1 = -\int_0^{x_1} (-k_1 x_1 - k_1 x_{1s} + mg \sin \theta_o) \hat{i}_1 \cdot d\vec{x}_1 = \int_0^{x_1} k_1 x_1 dx_1 = \frac{1}{2} k_1 x_1^2$$

$$V = V_{k_2} + V_{wheel} = \frac{1}{2} k_2 x^2 + \frac{1}{2} k_1 x_1^2$$

$$F_{spring} = K_1(x_{1static} + x_1)$$



Next find the kinetic energy T. This problem has two rigid bodies. The total kinetic energy is the sum of the kinetic energies of each rigid body.

For that part due to the rolling of the wheel, it is helpful to identify that this is a planar motion problem for which the axis of rotation is a principal axis of the wheel and it is perpendicular to the

plane of the translational motion of the system.

$$T = T_{cart} + T_{roller}$$

$$T_{cart} = \frac{1}{2} m_o \dot{x}^2$$

The kinetic energy of the roller involves both translation and rotation. The

appropriate equation to use is
$$T_{roller} = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} [I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + I_{zz} \omega_z^2]$$

$$\text{where } \omega_x = \omega_y = 0, \text{ and } \omega_z = \dot{\theta} = -\dot{x}_1 / R$$

For a uniform disk
$$I_{zz/G} = \frac{1}{2} m R^2$$

The remaining work is to find $v_{G/O}$ the velocity of the center of mass of the wheel relative to the inertial frame. Let G be the center of mass of the wheel.

Then the velocity of G in the inertial frame is given by

$$\vec{v}_{G/O} = \vec{v}_{C/O} + \vec{v}_{G/C} = \dot{x} \hat{i} + \dot{x}_1 \hat{i}_1 = \dot{x} \hat{i} + \dot{x}_1 (\cos \theta_o \hat{i} - \sin \theta_o \hat{j})$$

$$\vec{v}_{G/O} \cdot \vec{v}_{G/O} = \dot{x}^2 + \dot{x}_1^2 + 2\dot{x}\dot{x}_1 \cos \theta_o$$

The expression for the total kinetic energy of the system may now be obtained.

The expression for the total kinetic

$$T = \frac{1}{2}(m_o + m)\dot{x}^2 + \frac{1}{2}m[\dot{x}_1^2 + 2\dot{x}\dot{x}_1 \cos \theta_o] + \frac{1}{2}m\frac{R^2}{2}\dot{\theta}^2$$

$$T = \frac{1}{2}(m_o + m)\dot{x}^2 + \frac{1}{2}m[\dot{x}_1^2 + 2\dot{x}\dot{x}_1 \cos \theta_o] + \frac{1}{2}m\frac{\dot{x}_1^2}{2}$$

$$T = \frac{1}{2}(m_o + m)\dot{x}^2 + \frac{1}{2}m\left[\frac{3}{2}\dot{x}_1^2 + 2\dot{x}\dot{x}_1 \cos \theta_o\right]$$

b) Find the equations of motion using Lagrange equations, which may be stated as:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j \quad \text{where } L=T-V. \text{ For purely mechanical systems that have only}$$

springs and gravity as potential forces, the Lagrange equations may be more usefully written as follows:

$$\frac{d}{dt} \left(\frac{\partial(T)}{\partial \dot{q}_j} \right) - \frac{\partial(T)}{\partial q_j} + \frac{\partial(V)}{\partial q_j} = Q_j, \text{ where the } q_j \text{ are the generalized coordinates. This}$$

equation must be applied once for each generalized coordinate in the problem. In this problem, there are two generalized coordinates, x and x_1 . T and V are functions of x , \dot{x} , x_1 , and \dot{x}_1 . For the current problem there are no external non-conservative forces so the right hand side of the above equation is zero. Numbering the terms on the LHS as 1, 2 and 3 in order of appearance, the three terms may be evaluated in a systematic way.

Begin with the generalized coordinate x :

Term 1:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial(T)}{\partial \dot{x}} \right) &= \frac{d}{dt} \frac{\partial}{\partial \dot{x}} \left(\frac{1}{2}(m_o + m)\dot{x}^2 + \frac{1}{2}m\left[\frac{3}{2}\dot{x}_1^2 + 2\dot{x}\dot{x}_1 \cos \theta_o\right] \right) = \frac{d}{dt} ((m_o + m)\dot{x} + m\dot{x}_1 \cos \theta_o) \\ &= (m_o + m)\ddot{x} + m\ddot{x}_1 \cos \theta_o \end{aligned}$$

$$\text{Term 2: } \frac{\partial(T)}{\partial x} = 0$$

$$\text{Term 3: } \frac{\partial(V)}{\partial x} = k_2 x$$

Summing terms 1, 2 and 3 yields: $(m_o + m)\ddot{x} + m\ddot{x}_1 \cos \theta_o + k_2 x = 0$, which is the first equation of motion.

Turning to the second application of Lagrange, compute the three terms using the second generalized coordinate x_1 .

Term 1:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{x}_1}\right) &= \frac{d}{dt}\frac{\partial}{\partial\dot{x}_1}\left(\frac{1}{2}(m_o+m)\dot{x}^2 + \frac{1}{2}m\left[\frac{3}{2}\dot{x}_1^2 + 2\dot{x}\dot{x}_1\cos\theta_o\right]\right) = \frac{d}{dt}\left[\frac{3}{2}m\dot{x}_1 + m\dot{x}\cos\theta_o\right] \\ &= \frac{3}{2}m\ddot{x}_1 + m\ddot{x}\cos\theta_o\end{aligned}$$

Term 2 is zero

and Term 3 is : $\frac{\partial(V)}{\partial x_1} = k_1 x_1$

Summing terms 1, 2 and 3 yields: $\frac{3}{2}m\ddot{x}_1 + m\ddot{x}\cos\theta_o + k_1 x_1 = 0$, which is the second equation of motion.

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