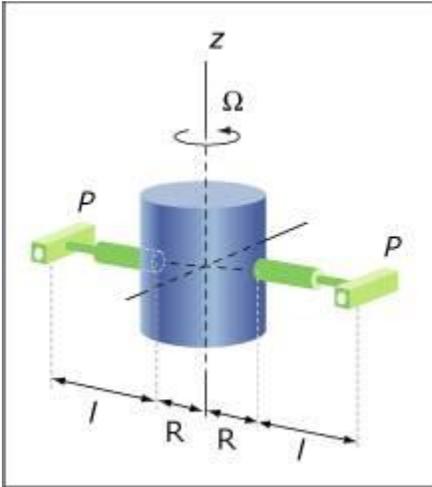


## 2.003 Engineering Dynamics

### Problem Set 3--solutions

#### Problem 1:

- a) A satellite is circling the earth. It spins at a constant angular rate  $\Omega = 0.05(\text{rads/s})\hat{k}$ . Two sensitive science instruments, indicated by 'P' in the figure, are symmetrically attached at the ends of extendable booms. Initially the booms are 1.2m in radius. Once the satellite is deployed in orbit the booms extend radially by a variable length L. As the booms are extended an internal mechanism is used to maintain the constant rotation rate of the satellite. The length L is varied from zero to 3 m. The maximum total acceleration to which the sensitive experiment modules may be subjected is  $0.011 \text{ m/s}^2$ . Determine the maximum allowable boom extension rate.



The solution to this problem requires an application of Euler's law governing the rotational dynamics of rigid bodies, as shown below, where C is the center of mass of the rigid body and O is the origin of the inertial reference frame used in the problem. In this problem the choice of the placement of the inertial frame requires careful consideration and will be addressed shortly. B in the equation is the point with respect to which the angular momentum is defined. That will turn out to be the center

of mass in this problem. In many problems the second term in Euler's equation is zero. A common example is when B is fixed in the inertial frame, then  $\mathbf{v}_{B/O}=0$ .

$$\sum_i \boldsymbol{\tau}_{i/B} = \left( \frac{d\mathbf{H}_{i/B}}{dt} \right)_{O_{xyz}} + \mathbf{v}_{B/O} \times \mathbf{P}_{C/O}$$

In this problem the point B is at the center of mass of the satellite. Therefore the second term in the equation above is zero, because the cross product term is always zero when the angular momentum is computed with respect to the center of mass. This is because the direction of the velocity of the center of mass is the same as the direction of the linear momentum. Noting that point B and the center of mass are the same, the equation above simplifies to:

$$\sum_i \tau_{i/C} = \left( \frac{dH_{/C}}{dt} \right)_{o_{xyz}} = 0$$

The equation is equal to zero because there are no external torques applied to the satellite. If the time rate of change of the angular momentum of the satellite with respect to the center of mass is zero then the angular momentum with respect to the center of mass must be a constant.

The problem statement specifies that the angular rotation rate of the instrument packages is a constant. As the booms are extended, the contribution from the booms and the instrument packages to the mass moment of inertia of the satellite with respect to an axis of rotation passing through the center of mass must increase. At constant rotation rate this would ordinarily result in an increase in the total angular momentum.

In order for the angular momentum of the whole satellite to remain constant there must be some device inside of the satellite whose change in angular momentum is equal and opposite to that caused by the extension of the instrument packages. The result is that the total angular momentum of the satellite remains constant. An example device would be a rotor which spins in the opposite direction to the rotation rate of the instrument packages. As the instrument packages are extended the rotation rate of the internal rotor increases in the direction opposite to the rotation rate of the instrument packages. The total angular momentum of the internal rotor plus the rest of the satellite remains constant.

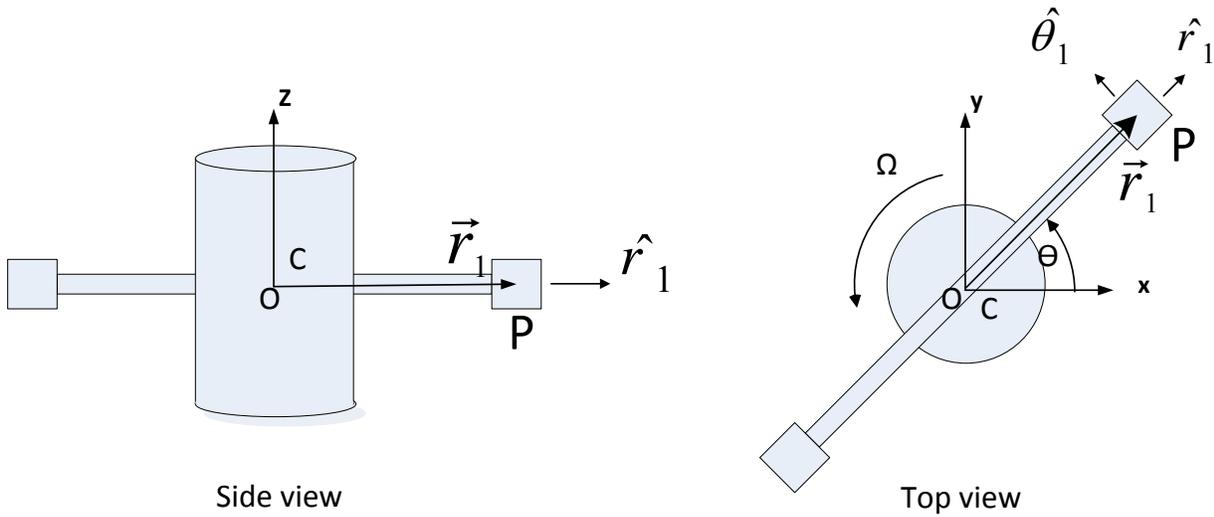
It remains to find the maximum velocity of the extension of the instrument package arms such that the total acceleration does not exceed the limit given,  $0.011 \text{ m/s}^2$ . Kinematic expressions are needed to express the velocity and the acceleration of the instrument packages with respect to the center of mass of the satellite.

The problem statement as given is somewhat ambiguous. It does not make clear whether or not the acceleration of satellite due to its travel in earth orbit is to be considered. In order for the satellite to stay in orbit, the weight of the satellite must be exactly equal to the mass of the satellite times the centripetal acceleration due to circular motion about the earth. At an altitude of 100 miles in a low earth orbit the centripetal acceleration is approximately  $0.95G=9.4\text{m/s}^2$ . This is far in excess of the  $0.011 \text{ m/s}^2$  allowed in the problem statement. Why is it meaningful to focus on this small quantity rather than the much larger acceleration resulting from the orbital motion?

The reason that the acceleration due to orbital motion may be ignored is because it results in no local stresses on the instrument package. The mass times the centripetal acceleration of the instrument package as it circles the earth is exactly equal to the gravitational force of the earth on instrument package. A passenger on the satellite would feel weightless. In this case the orbital motion leads to no stresses on the instrument package. However, the acceleration of the package relative to the center of mass of the satellite will lead to local forces and stresses necessary to

make the package move in the way specified.

Thus the velocity and acceleration of the instrument package may be computed with respect to an inertial frame located at the center of mass, but not rotating with the satellite. The inertial frame  $O_{xyz}$  is shown in the figure below. The velocity and the acceleration may be computed using either polar coordinates or the full vector expression for velocity and acceleration. These expressions are given below for polar coordinates, where the vector  $\vec{r}_1 = r\hat{r}_1$  is the position vector of one instrument package with respect to the center of mass. The unit vector  $\hat{r}_1$  is aligned with the boom supporting one of the packages, as shown in the figure.



$$\vec{r}_1 = r\hat{r}_1, \text{ where } r_{\max} = 1.2\text{m} + 3.0\text{m} = 4.2\text{m}$$

$$\vec{V}_{P/O} = \left[ \frac{d\vec{r}_1}{dt} \right]_{O_{xyz}} = \dot{r}\hat{r}_1 + r\dot{\theta}_1\hat{\theta}_1, \text{ where } \Omega = \dot{\theta}_1 = .05 \frac{r}{s}$$

$$\vec{a}_{P/O} = [\ddot{r}_1 - r\dot{\theta}_1^2]\hat{r}_1 + [r\ddot{\theta}_1 + 2\dot{r}\dot{\theta}_1]\hat{\theta}_1 = [0 - r\Omega^2]\hat{r}_1 + [0 + 2\dot{r}\Omega]\hat{\theta}_1$$

$$\vec{a}_{P/O} = -(4.2\text{m})(.05 \frac{r}{s})^2 \hat{r}_1 + 2\dot{r}(.05 \frac{r}{s})\hat{\theta}_1, \text{ where } r = r_{\max} = 4.2\text{m}$$

$$|\vec{a}_{P/O}|^2 = [(4.2\text{m})(.05 \frac{r}{s})^2]^2 + [2\dot{r}(.05 \frac{r}{s})]^2 \leq [0.011 \frac{m}{s^2}]^2 \quad (a)$$

$$\Rightarrow \text{the maximum allowable } \dot{r} \leq 0.033 \frac{m}{s}$$

The computed acceleration has two terms. One depends on the radius  $r$  and the other on the velocity  $\dot{r}$ . The acceleration is greatest when both are maximum. The first term is the centripetal acceleration due to the rotation of the instrument package about the center of mass. The second

term is the Coriolis acceleration, which results in an increase in the angular momentum of the instrument package as the boom extends. The magnitude of the centripetal acceleration with respect to the center of mass of the satellite is maximum when the arm is at the maximum extension of 4.2 m. The two terms are orthogonal to one another and their magnitude is simply the square root of the sum of the squares of the two parts. The maximum acceleration magnitude will be achieved at maximum radius. Equation (a) above was solved for the allowable rate of extension of 0.033m/s. This is the end of the simplified problem, in which the orbital motion was ignore. For the curious, this addendum to the solution of problem 1 addresses how one would include the orbital motion.

If one wished to compute the total acceleration of the instrument package including the effects of orbital motion, then it would be best to employ the full 3D vector equation for acceleration of a point in a translating and rotating reference frame as given by the following expression. Note that the  $O_{xyz}$  frame would be an approximate inertial frame at the center of the earth. The  $A_{x_1y_1z_1}$  frame is fixed to the satellite at its center of mass. The point B would be at the location of the instrument package. The first term  $\mathbf{a}_{A/O}$  would be the centripetal acceleration of the satellite due to the orbital motion. In this problem, the 2<sup>nd</sup> and 3<sup>rd</sup> terms are zero. The 4<sup>th</sup> and 5<sup>th</sup> terms are those already computed that account for the centripetal and coriolis accelerations with respect to the center of mass of the satellite.

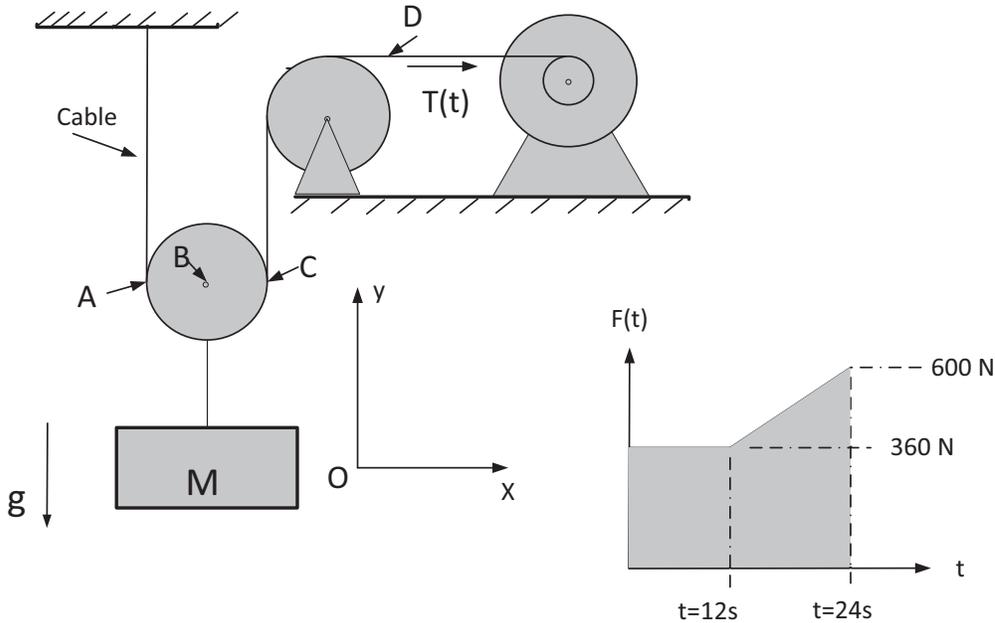
$$\vec{a}_{B/O} = \vec{a}_{A/O} + (\vec{a}_{B/A})_{/A_{x_1y_1z_1}} + \dot{\omega} \times \vec{r}_{B/A} + \vec{\omega}_{/O} \times \vec{\omega}_{/O} \times \vec{r}_{B/A} + 2\vec{\omega}_{/O} \times (\vec{V}_{B/A})_{/A_{x_1y_1z_1}}$$

$$\vec{a}_{B/O} = (1) + (2) + (3) + (4) + (5)$$

- (1) ≡ Centripetal acceleration of frame  $A_{x_1y_1z_1}$  with respect to the inertial frame  $O_{xyz}$ .
- (2) ≡ Relative acceleration of the point B as seen from within the moving frame.
- (3) ≡ Eulerian acceleration due to the angular acceleration of the rotating body.
- (4) ≡ Centripetal acceleration term due to the body's rotational velocity.
- (5) ≡ Coriolis acceleration due to translational velocity of an object at B relative to the rotating rigid body.

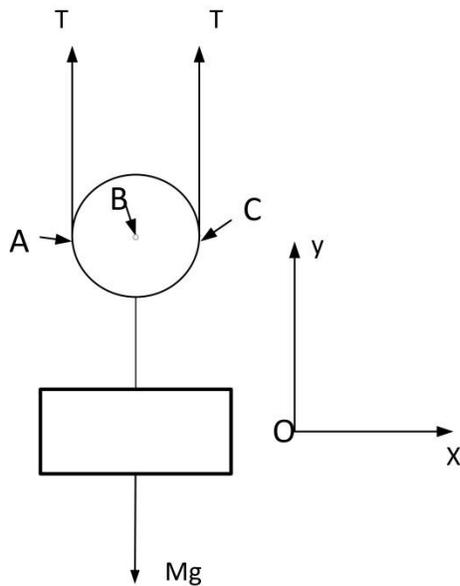
**Problem 2:**

The winch, shown in the figure, delivers a horizontal towing force, T(t), to its cable at D. The force varies with time as shown in the graph. Determine the speed of the 80-kg mass when ‘t’ reaches 24 s. Assume that at t=0, the mass has zero velocity. Also assume that the pulleys and cables are massless.



The solution to this problem requires finding an expression for the acceleration of the hanging mass. This is done by direct application of free body diagrams and Newton's second law. Use the

fact that the tension on a cable passing around a massless pulley remains constant, as can be proven by summing the torques about the axle and requiring the sum be zero, because the mass moment of inertia of the pulley is zero.



A free body diagram of system consisting of the mass and the pulley to which it is connected is shown. Application of Newton's 2<sup>nd</sup> law reveals that:

$$\sum_{Ext} F = \frac{dP}{dt} = M\bar{a} = M\dot{y}\hat{j} = (2T - Mg)\hat{j}$$

$$\therefore T = M(\ddot{y} + g) / 2$$

Since the tension as a function of time is known this equation may be solved for the acceleration and then integrated to give the velocity when t=24s.

$$T = M(\ddot{y} + g) / 2$$

$$\ddot{y} = \frac{2T}{M} - g$$

$$\dot{y} = \int_0^t \ddot{y} dt = \int_0^{12} \ddot{y} dt + \int_{12}^{24} \ddot{y} dt$$

In the interval from  $t=0$  to  $t=12$  s,  $\ddot{y} = \frac{2T}{M} - g = \frac{2 \times 360 \text{ N}}{80 \text{ kg}} - 9.81 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = -0.81 \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$

In the interval from  $t=12$  to  $t=24$  s,

$$T(t) = [360 \text{ N} + (t-12)20 \frac{\text{N}}{\text{s}}]$$

$$\ddot{y} = \frac{2T}{M} - g = \frac{2 \times [360 \text{ N} + (t-12)20 \frac{\text{N}}{\text{s}}]}{80 \text{ kg}} - 9.81 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = [-6.81 + 0.5t] \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$$

Integrating over the two segments of time reveals the following:

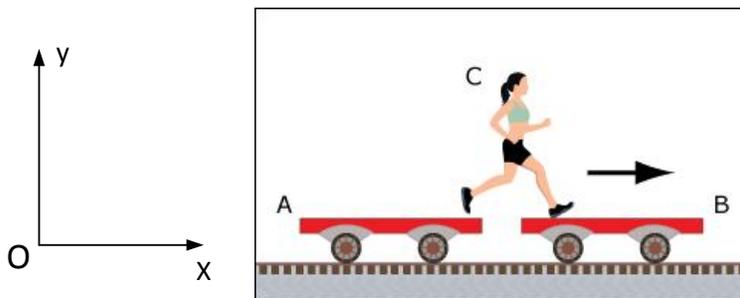
$$\dot{y} = \int_0^{12} \ddot{y} dt + \int_{12}^{24} \ddot{y} dt = \int_0^{12} [-0.81 \frac{\text{kg} \cdot \text{m}}{\text{s}^2}] dt + \int_{12}^{24} [-6.81 + 0.5t] dt$$

$$\dot{y} = [-0.81t]_0^{12} + [-6.82t + 0.25t^2]_{12}^{24} = 16.44 \frac{\text{m}}{\text{s}}$$

### Problem 3:

A 75-kg girl leaps from cart A with a horizontal velocity of 3 m/s measured relative to cart A. Carts A and B have the same mass of 50 kg and are originally at rest.

- Determine the velocity of cart A just after she jumps.
- If she then lands on cart B and comes to a stop relative to cart B, determine the velocity of cart B just after she lands on it.



Solution:

The solution to this problem is based on the conservation of linear momentum in the horizontal direction. Assuming no external friction forces in the horizontal direction act on either cart, then the total linear momentum of the girl and the cart is conserved in the horizontal direction.

The carts are designated A and B and the girl is designated C. Prior to jumping from cart A the initial linear momentum of the system is zero. After jumping the girl and cart A each have nonzero velocity. The total momentum of the system may be written as:

State 1: Before jumping,  $\mathbf{P}_{\text{total}}=0$ .

State 2: After leaping from cart A:

$$\sum_i \vec{P}_i = M_C \vec{v}_{C/O} + M_A \vec{v}_{A/O} = 0 \quad (\text{a})$$

You are given that  $\vec{v}_{C/A} = +3 \frac{m}{s} \hat{i}$ ;

Also known is the relation for velocity of a point in an inertial frame using the velocity expressed in a translating frame:

$$\vec{v}_{C/O} = \vec{v}_{A/O} + \vec{v}_{C/A} = \vec{v}_{A/O} + 3 \frac{m}{s} \hat{i} \quad (\text{b})$$

Substituting for  $\vec{v}_{C/O}$  from (b) into (a) yields

$$\begin{aligned} M_C (\vec{v}_{A/O} + 3 \frac{m}{s} \hat{i}) + M_A \vec{v}_{A/O} &= 0 \\ \Rightarrow \vec{v}_{A/O} &= (-3 \frac{m}{s} \hat{i}) \frac{M_C}{M_A + M_C} = -3 \frac{75\text{kg}}{125\text{kg}} \frac{m}{s} \hat{i} = -1.8 \frac{m}{s} \hat{i} \end{aligned}$$

State 3: After the girl lands on cart B, the total system momentum remains zero, as expressed in the next equation:

$$\sum_i \vec{P}_i = (M_C + M_B) \vec{v}_{B/O} + M_A \vec{v}_{A/O} = 0 \quad (\text{a})$$

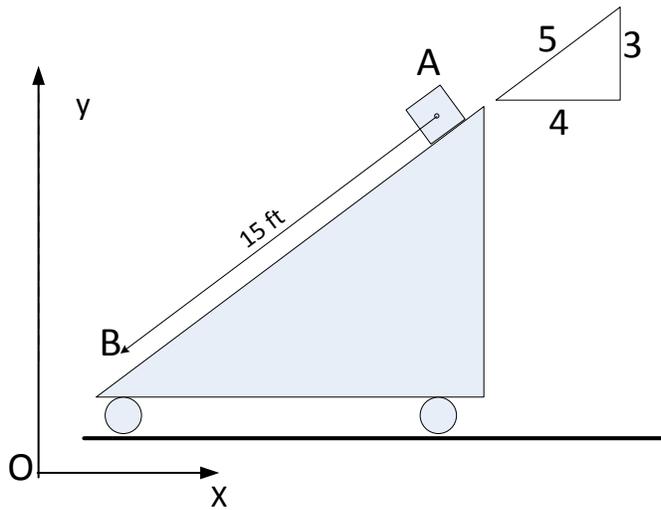
Given  $\vec{v}_{A/O} = -1.8 \frac{m}{s} \hat{i}$

$$\Rightarrow \vec{v}_{B/O} = -\frac{M_A}{M_B + M_C} \vec{v}_{A/O} = \frac{50\text{kg}}{125\text{kg}} 1.8 \frac{m}{s} \hat{i} = +0.72 \frac{m}{s} \hat{i}$$

#### Problem 4:

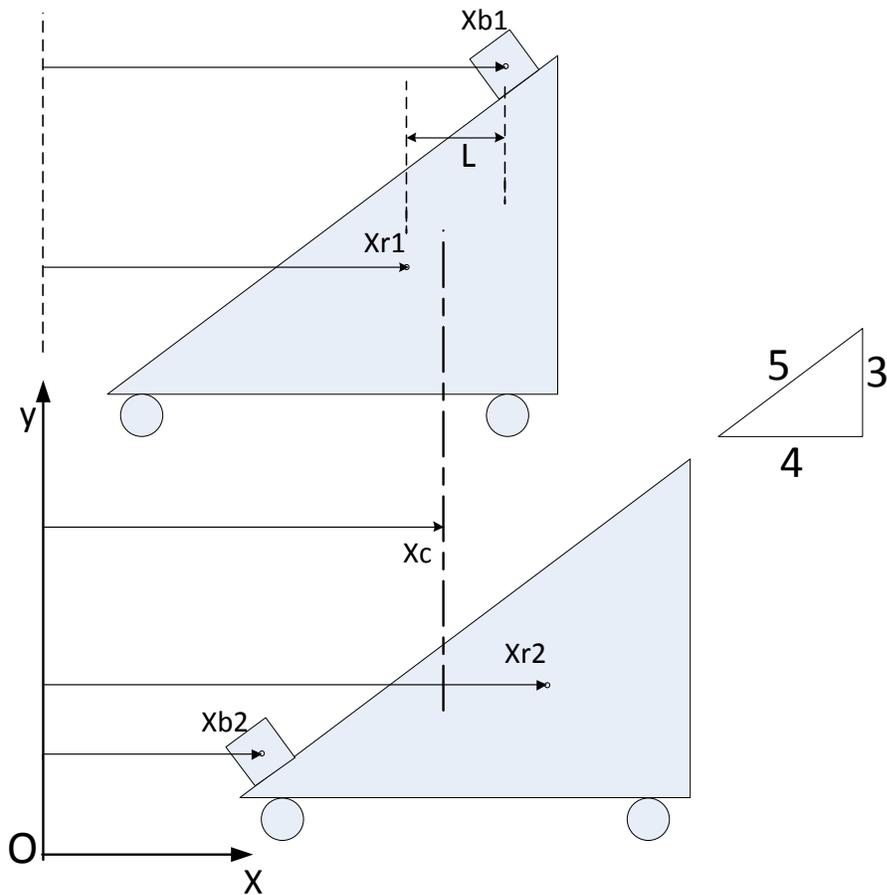
A ramp rolls without friction on a horizontal floor. It has a weight of 120 lb. If an 80-lb crate is released from rest at point A near the top of the ramp, determine the distance the ramp moves

with respect to the floor when the crate slides 15 ft down the ramp to point B.



Solution:

In the absence of friction between the ramp and the floor in the X direction the system consisting of the ramp and box obeys conservation of linear momentum. This will be true whether or not there is friction between the box and the ramp, because that friction is an internal force of the whole system which includes the box and the ramp. If the box and the ramp are initially at rest then the velocity of the center of mass of the system is zero. In the absence of external forces in the x direction the center of mass of the system must not move in the X direction. As the box slides a component of its motion is in the minus X direction. At the same time the ramp will move in the positive X direction so as to keep the center of mass motionless. The box and ramp are shown in the initial and final positions in the figure below.



Let  $X_{b1}$  and  $X_{b2}$  be the initial and final X coordinates of the center of mass of the box. Let  $X_{r1}$  and  $X_{r2}$  be the initial and final X coordinates of the position center of mass of the ramp. Let  $X_c$  be the coordinate of the center of mass of the box and ramp treated as a system. Using the definition of the center of mass of the system in the X direction:

$$(M_b + M_r) X_c = M_b X_{b1} + M_r X_{r1} = M_b X_{b2} + M_r X_{r2}$$

$$\Rightarrow M_b (X_{b2} - X_{b1}) + M_r (X_{r2} - X_{r1}) = 0 \quad (a)$$

Let the initial distance between the centers of mass of the box and the ramp be given by  $L$ . After the box slides down the ramp this distance changes by 12 feet, which is the horizontal component of the 15 foot displacement down the ramp. Expressed mathematically this becomes:

Initially

$$X_{b1} - X_{r1} = L \text{ (ft)} \quad (\text{b})$$

After sliding 12 feet to the left the appropriate expression is

$$X_{b2} - X_{r2} = L - 12 \quad (\text{c})$$

Subtracting equation (b) from (c) leads to

$$(X_{b2} - X_{b1}) - (X_{r2} - X_{r1}) = -12 \text{ (ft)} \quad (\text{d})$$

Solving for  $(X_{b2} - X_{b1})$  yields

$$(X_{b2} - X_{b1}) = -12 \text{ (ft)} + (X_{r2} - X_{r1}) \quad (\text{e})$$

Substituting (e) into (a) and solving for  $(X_{r2} - X_{r1})$  yields:

$$(X_{r2} - X_{r1}) = 12 \frac{M_b}{M_b + M_r} \text{ (ft)} = 12 \left( \frac{80 \text{ lbs}}{200 \text{ lbs}} \right) = 4.8 \text{ ft}$$

The ramp moves 4.8 feet to the right, while the center of mass of the system does not move at all.

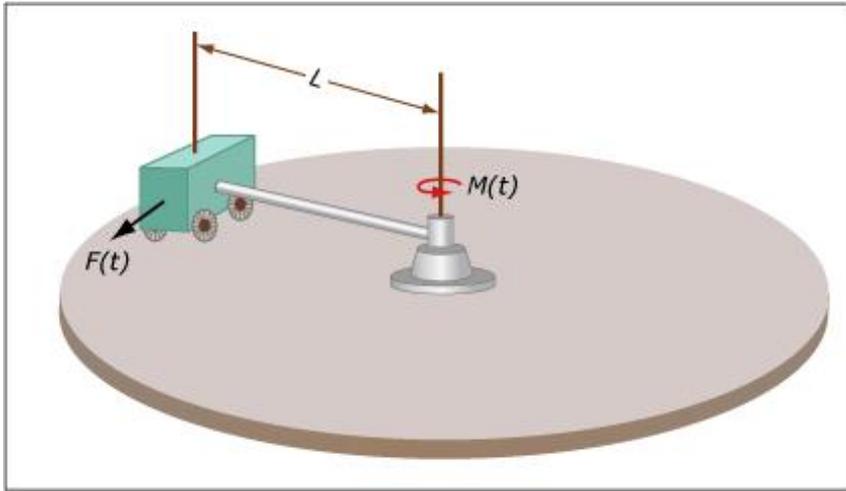
### Problem 5

A massless arm rotates about a vertical axis. At the outer end of the arm is a cart. For the purpose of this problem consider the cart to be a concentrated particle. A moment (torque),  $M(t)$  is applied to the arm at the axis of rotation, given by  $M(t) = 30t^2 (N - m)$ . In addition an external force is applied to the mass in the tangential direction. The force is given by  $F(t) = 15t(N)$ . The force and moment are applied beginning at  $t=0$ .

(a) Find an expression for the angular momentum of the cart with respect to the point on the axis of rotation where the rod attaches to the pivot.

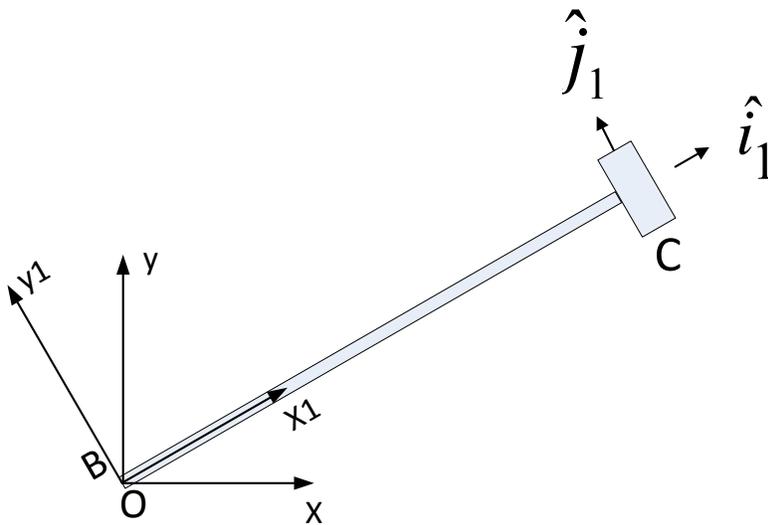
(b) At  $t=5$  seconds the external driving forces and moments are turned off, leaving the arm and cart to coast at a constant rotation rate. Find an expression for  $\vec{P}$ , the linear momentum of the cart after the force and moment are turned off.

(c) Compute the time derivative of  $\vec{P}$  with respect to the Oxyz inertial frame. By Newton's 2<sup>nd</sup> law the result must be the force applied to the mass. Explain the physical meaning of this force.



Solution:

- a) The solution is based on the concept of impulse of angular momentum. It is appropriate to begin with Euler's law for the angular momentum of a single particle as given below, where  $O_{xyz}$  refers to a fixed inertial frame. The  $B_{x_1y_1z_1}$  frame is attached to the rotating system, such that the  $x_1$  axis is along the rod. The origins of both reference frames,  $O$  and  $B$ , are both at the location of the fixed axis of rotation, as shown in the figure. The mass is located at point  $C$ .



$$\sum_i \tau_{i/B} = \left( \frac{d}{dt} (\vec{r}_{C/B} \times \vec{P}) \right)_{O_{xyz}} = \left( \frac{d}{dt} (\vec{r}_{C/B} \times m\vec{v}_{C/O}) \right)_{O_{xyz}} = \left( \frac{d}{dt} (L\hat{i}_1 \times m(\boldsymbol{\omega}\hat{k} \times L\hat{i}_1)) \right)_{O_{xyz}} = \left( \frac{d}{dt} (h_{/B}) \right)_{O_{xyz}}$$

where,  $h_{/B} = mL^2\boldsymbol{\omega}\hat{k}$  and

$$\left( \frac{d}{dt} (h_{/B}) \right)_{O_{xyz}} = mL^2\dot{\boldsymbol{\omega}}\hat{k}$$

Angular momentum is indicated here by a lower case 'h' to indicate it is for a single particle.

Upper case H will be used to indicate the angular momentum for extended rigid bodies. The left hand side of this expression is the sum of the external torques with respect to point B. These torques result from the applied external force, F(t), and moment, M(t), as shown below:

$$\sum_i \tau_{i/B} = M(t) + \vec{r} \times F(t) = 30t^2\hat{k} + L\hat{i}_1 \times 15t\hat{j}_1 = [30t^2 + 15Lt]\hat{k}(N - m)$$

These external torques are only in the z direction. This is a fixed axis planar motion problem. The angular momentum of the system and its derivative are both in the z direction, perpendicular to the plane of the motion. The change in the total angular momentum of the system may be obtained by integrating the applied torques with respect to time to obtain the impulse of angular momentum.

$$h_{B,final} = h_{B,initial} + \int \sum_i \tau_{i/B} dt = h_{B,initial} + \int_0^t [30t^2 + 15Lt]\hat{k}(N - m) dt$$

$$h_{B,initial} = 0$$

$$h_{B,final} = \int_0^t [30t^2 + 15Lt]\hat{k}(N - m) dt = (10t^3 + 7.5t^2)((N - m - s)) \Big|_0^t = 1250 + 187.5 = 1437.5(N - m - s)$$

$$h_{B,final} = mL^2\boldsymbol{\omega}\hat{k} = 1437.5(N - m - s)\hat{k}$$

To find  $\boldsymbol{\omega}$ , let L=4m and m=100kg

then

$\boldsymbol{\omega} = 0.90\hat{k}$  radians/second, which is constant for  $t \geq 5$  seconds.

b) For t greater than 5 seconds the linear momentum is given by

$\vec{P} = m\vec{v}_{C/O} = m(\boldsymbol{\omega}\hat{k} \times L\hat{i}_1) = mL\boldsymbol{\omega}\hat{j}_1 = 359\hat{j}_1(N - m/s)$  In this expression for the linear momentum, all parameters are constant except for the unit vector  $\hat{j}_1$ . It changes direction with time.

c) The time derivative of the linear momentum is given by

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(mL\omega\hat{j}_1) = mL\omega\frac{d}{dt}(\hat{j}_1) = -mL\omega^2\hat{i}_1(N)$$

because  $\frac{d}{dt}(\hat{j}_1) = -\omega\hat{i}_1$

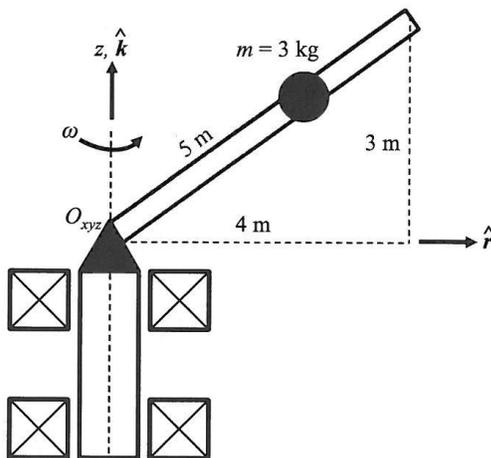
This is the force that the arm must exert on the cart to make the cart move at constant speed in a circle. It is the mass times the centripetal acceleration of the cart.

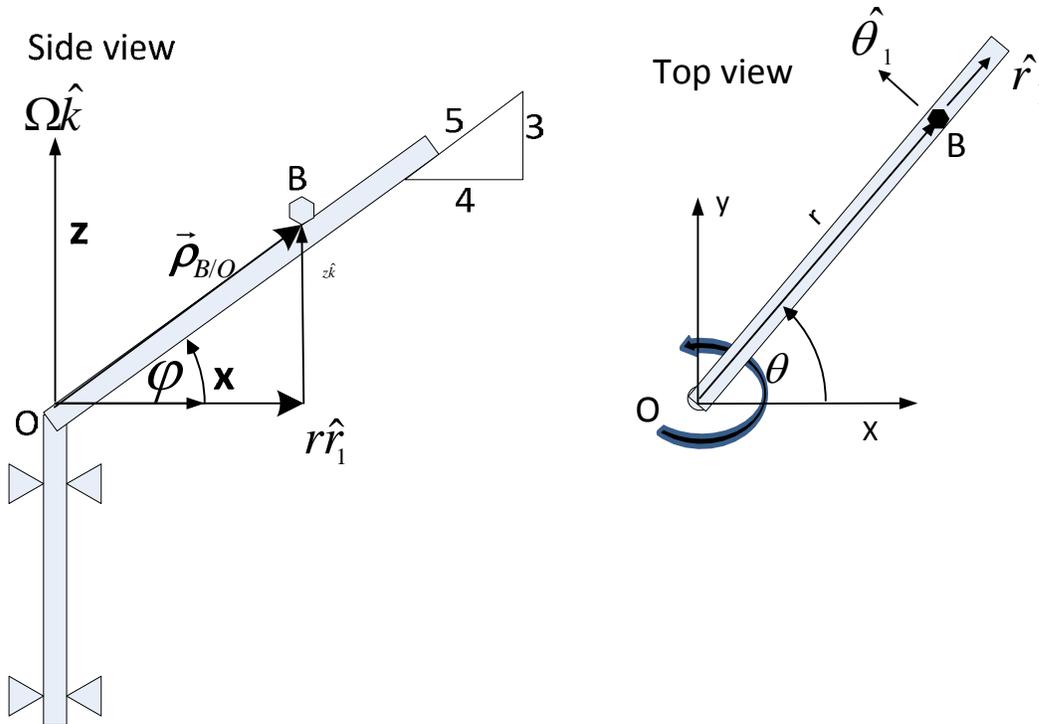
Newton's second law works very well in this unusual situation.

### Problem 6

A 3 kg monkey runs up the shaft shown in the figure. The shaft rotates at a constant rate of 2.0 rad/s. The speed of the monkey running along the shaft is 1.5 m/s.

- Find expressions for the velocity and acceleration of the monkey with respect to a fixed inertial frame,  $Oxyz$ , located at the point the inclined shaft connects to the vertical rotating shaft.
- Treating the monkey as a simple particle, find an expression for the angular momentum of the monkey with respect to the origin at  $O$ . What is the direction of the angular momentum vector? Draw an arrow that begins at  $O$  to represent the angular momentum vector,  $h_{/O}$ .
- Compute the time rate of change of  $h_{/O}$  with respect to the  $Oxyz$  frame. It will have three vector components or terms. Give a physical interpretation to the meaning of each term.





Solution:

a). The first requirement is to find an expression for the velocity and acceleration of the monkey expressed in the  $O_{xyz}$  frame. Let the position of the monkey be designated by point B.

Cylindrical coordinates  $(r_1, \theta, z)$  are convenient in this problem. The position vector for the monkey,  $\vec{\rho}_{B/O}$ , has components in the  $\hat{r}_1$  and  $\hat{k}$  directions such that  $\vec{\rho}_{B/O} = r\hat{r}_1 + z\hat{k}$

$$\vec{V}_{B/O} = \left[ \frac{d\vec{\rho}_{B/O}}{dt} \right]_{O_{xyz}} = (\dot{r}\hat{r}_1 + \dot{z}\hat{k}) + \vec{\omega} \times \vec{\rho}_{B/O} = (\dot{r}\hat{r}_1 + \dot{z}\hat{k}) + \Omega\hat{k} \times (r\hat{r}_1 + z\hat{k}) \quad (a)$$

$$= (\dot{r}\hat{r}_1 + \dot{z}\hat{k}) + r\Omega\hat{\theta}_1, \text{ where } \Omega = \dot{\theta} = 2.0 \frac{\text{rad}}{\text{s}}, r = 5\text{m} \frac{4}{5} = 4\text{m}, \dot{r} = (1.5\text{m/s}) \frac{4}{5} = 1.2 \text{ m/s},$$

$$\text{and } \dot{z} = (1.5\text{m/s}) \frac{3}{5} = 0.9\text{m/s}.$$

To convert to unit vectors in the inertial frame,  $O_{xyz}$ , use the familiar substitution

$$\hat{r}_1 = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}$$

$$\hat{\theta}_1 = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j}, \text{ where } \theta = \Omega t, \text{ leading to the following expressions.}$$

$$\vec{V}_{B/O} = (\dot{r}\hat{r}_1 + \dot{z}\hat{k}) + r\dot{\theta}\hat{\theta}_1 = (\dot{r}(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}) + \dot{z}\hat{k} + r\dot{\theta}(-\sin(\theta)\hat{i} + \cos(\theta)\hat{j}))$$

$$\vec{V}_{B/O} = (\dot{r} \cos(\theta) - r\dot{\theta} \sin(\theta))\hat{i} + (\dot{r} \sin(\theta) + r\dot{\theta} \cos(\theta))\hat{j} + \dot{z}\hat{k}$$

To find the acceleration of the monkey, apply the following expression.

$$\vec{a}_{B/O} = \left[ \frac{d\vec{V}_{B/O}}{dt} \right]_{O,xyz}$$

Taking a time derivative of  $\vec{V}_{B/O}$  with respect to the fixed inertial frame leads to:

$$\vec{a}_{B/O} = \ddot{z}\hat{k} + [\dot{r} - r\dot{\theta}^2]\hat{r}_1 + [r\ddot{\theta} + 2\dot{r}\dot{\theta}]\hat{\theta}_1$$

Given that the speed of the monkey and the rotation rate are constant,

$$\dot{r} = \dot{\theta} = \dot{z} = 0$$

$$\vec{a}_{B/O} = -r\dot{\theta}^2\hat{r}_1 + 2\dot{r}\dot{\theta}\hat{\theta}_1 = -r\Omega^2\hat{r}_1 + 2\dot{r}\Omega\hat{\theta}_1$$

There are two terms in the acceleration felt by the monkey. The first is the centripetal acceleration which is associated with circular motion. The second is the coriolis acceleration necessary to provide the increase in the angular momentum of the monkey as he moves further from O.

Again applying the conversion from polar coordinate unit vectors to Cartesian unit vectors leads to the following expressions in the inertial frame. As we shall see in parts b and c it is much easier to do the vector math in terms of the rotating unit vectors.

$$\vec{a}_{P/O} = -r\Omega^2\hat{r}_1 + 2\dot{r}\Omega\hat{\theta}_1 = -r\Omega^2(\cos(\Omega t)\hat{i} + \sin(\Omega t)\hat{j}) + 2\dot{r}\Omega(-\sin(\Omega t)\hat{i} + \cos(\Omega t)\hat{j})$$

b). Next an expression is required for the angular momentum of the monkey. In this problem treat the monkey as a simple particle. Angular momentum with respect to the origin, O, is given by:

$$\begin{aligned} h_{/O} &= \vec{\rho}_{B/O} \times P_{/O} = (r\hat{r}_1 + z\hat{k}) \times m\vec{V}_{B/O} \\ &= m(r\hat{r}_1 + z\hat{k}) \times [(\dot{r}\hat{r}_1 + \dot{z}\hat{k}) + r\Omega\hat{\theta}_1] = m(r\hat{r}_1 + z\hat{k}) \times r\Omega\hat{\theta}_1 = m\vec{\rho}_{B/O} \times r\Omega\hat{\theta}_1 \end{aligned}$$

where it has been noted that  $(r\hat{r}_1 + z\hat{k}) \times (\dot{r}\hat{r}_1 + \dot{z}\hat{k}) = 0$ , because the position vector to the monkey and the speed of the monkey relative to the arm are in the same direction. Therefore,

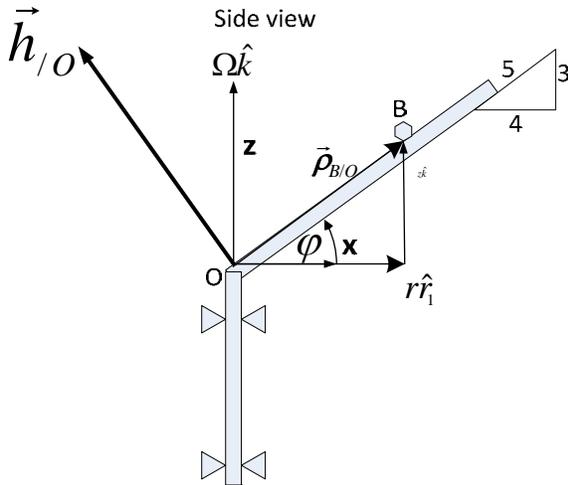
$$h_{/O} = m(r\hat{r}_1 + z\hat{k}) \times r\Omega\hat{\theta}_1 = m\vec{\rho}_{B/O} \times r\Omega\hat{\theta}_1 = mr^2\Omega\hat{k} - m zr\Omega\hat{r}_1$$

We could have anticipated that the direction that the angular momentum vector points must be perpendicular to both  $\vec{\rho}_{B/O}$  and  $P_{/O}$ . This is because they are the two terms in the cross product that define  $\mathbf{h}_{/O}$ . The direction of  $\mathbf{h}_{/O}$  is perpendicular to the arm that the monkey is climbing, as in the sketch below.

This is a remarkable result in that unlike most problems you have encountered before in more introductory subjects on mechanics, the angular momentum vector and the rotation rate vector

are not parallel. This has significant physical meaning. Any time a rigid body is rotating about a fixed axis and the computed angular momentum vector does not point in the same direction as the rotation rate vector, the rotating body has a dynamic imbalance. The body will require the application of additional external moments to hold it in place as it spins. These additional torques usually do no work but they are necessary to keep the body from collapsing as it spins.

They do produce bending moments in the structure and can lead to failure of the device if they are too large.



c). The final part of this problem is to compute the time rate of change of the angular momentum,  $\mathbf{h}_{/O}$ . It has already been stated that some unexpected moments are required to restrain the body as it rotates. Computing the time rate of change of the momentum will reveal exactly what these unknown moments are. From part b) the angular momentum is given by

$$\vec{h}_{/O} = mr^2\Omega\hat{k} - m_zr\Omega\hat{r}_1$$

We need to take its derivative. This vector changes direction with time because the unit vector  $\hat{r}_1$  changes direction with time. There is a general formula for the time derivative of a rotating vector  $\mathbf{P}$ :

$$\frac{d\vec{P}}{dt} = \left( \frac{\delta\vec{P}}{\delta t} \right)_{\text{w/o rotation}} + \vec{\omega} \times \vec{P}$$

The first term is the time rate of change of the vector, ignoring the effect of the rotation of the vector. The second term accounts for the effect of rotation. For example, in part a) of this problem an expression was found for the velocity of the monkey as the time derivative of the rotating position vector  $\vec{\rho}_{B/O}$ . The first term above would yield the velocity of the monkey relative to the arm. The second term gives the contribution to the velocity of the monkey due only to the rotation. Applying this formula to the angular momentum expression above yields:

$$\sum_i \tau_{i/O} = \frac{d\vec{h}_{i/O}}{dt} = \left( \frac{\delta \vec{h}_{i/O}}{\delta t} \right)_{\text{w/o rotation}} + \vec{\omega} \times \vec{h}_{i/O}$$

$$\sum_i \tau_{i/O} = \frac{dh_{i/O}}{dt} = 2mr\dot{r}\Omega\hat{k} + (-m\dot{z}r\Omega - mz\dot{r}\Omega)\hat{r}_1 + \Omega\hat{k} \times (mr^2\Omega\hat{k} - m\dot{z}r\Omega\hat{r}_1)$$

$$\sum_i \tau_{i/O} = 2mr\dot{r}\Omega\hat{k} + (-m\dot{z}r\Omega - mz\dot{r}\Omega)\hat{r}_1 - m\dot{z}r\Omega^2\hat{\theta}_1$$

Euler's law for angular momentum says that for bodies rotating about fixed axes, the time rate of change of the angular momentum must be equal to the sum of all of the external torques acting on the body. The above result tells us that there must be three torque components acting on this object at point O, one in each of the unit vector directions.

Are these real torques and do they have real physical meanings. For certain they do. The first term on the right hand side is a torque about the axis of rotation. This is the torque required to increase the angular momentum of the monkey as it moves its mass further away from the axis of rotation. This term does work, which results in an increase in the kinetic energy of the monkey.

The third term is the torque required to counter the moment created by the centrifugal force of the mass of the monkey acting on a moment arm z in length. This term leads to stresses but does not work.

The second term accounts for the moment created by the coriolis force which would lead to twisting of the arm about the  $\mathbf{r}_1$  axis. This moment causes stresses as well but does no work.

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