2.003SC
Recitation 5 Notes: Torque and Angular Momentum, Equations of Motion for Multiple Degree-of-Freedom Systems

Things To Remember

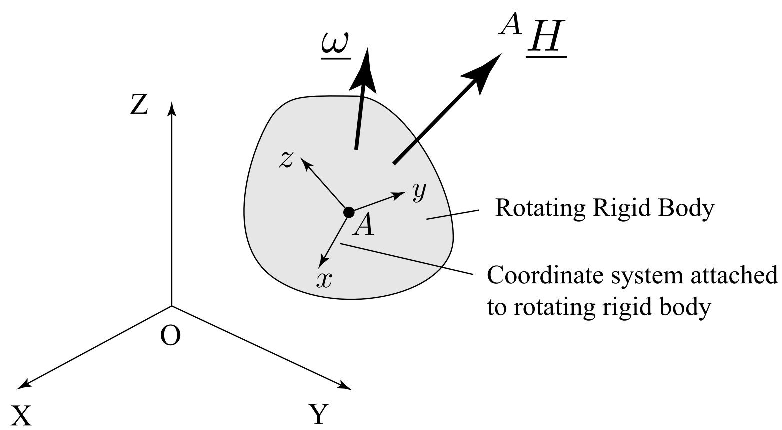
- Frame $Axyz$ is attached to (and rotates with) the rotating rigid body
- If A is chosen to be EITHER

  fixed in space ($\underline{v}_A = 0$) OR

  the rigid body's center of mass ($\underline{v}_A = \underline{v}_G$)

  then

$$\Sigma {}^A\underline{\tau}_{ext} = \frac{d\,{}^A\underline{H}}{dt}$$



- Both ${}^A\underline{H}$ and $\underline{\omega}$ are expressed in the unit vectors of $Axyz$
- The angular momentum can be expressed as

$$[{}^AH] = [I_A][\omega]$$

- If ${}^A\underline{H}$ and $\underline{\omega}$ are <u>not aligned</u>, then

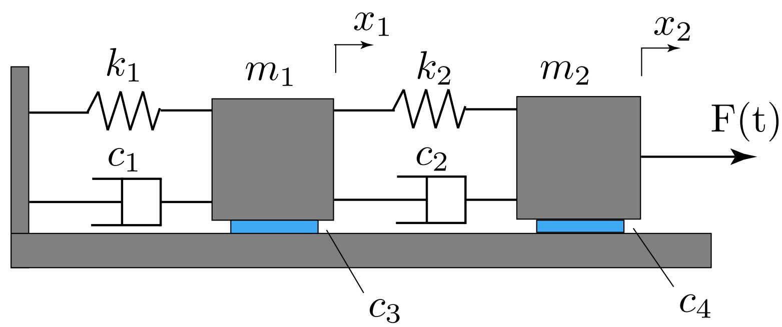
  there are non-zero off-diagonal terms in $[I_A]$, and

  there are components of $\Sigma {}^A\underline{\tau}_{ext}$ not in the direction of $\underline{\omega}$

Equations of Motion for Multiple Degree-of-Freedom Systems

Problem Statement

The figure below shows a system of two masses, m_1 and m_2, two springs, k_1 and k_2 , and four viscous damping elements, c_1, c_2, c_3, c_4, as well as an external force $F(t)$ acting on the second mass.



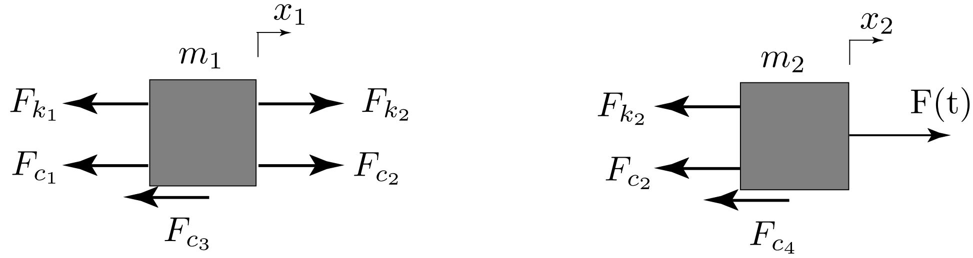
Note that the damping elements c_3 and c_4 act between the masses and the ground.

- Draw the system's Free Body Diagrams
- Derive the system's Equations of Motion

Solution

Free Body Diagrams

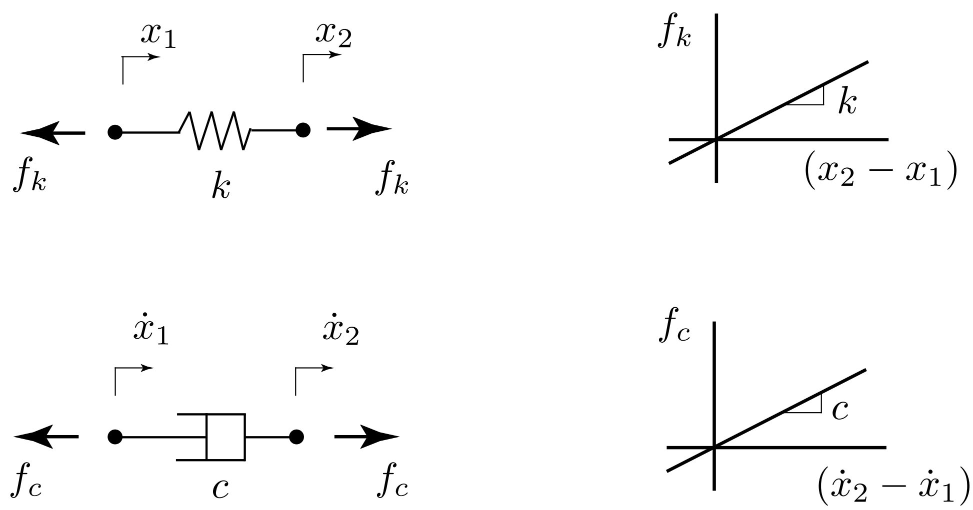
Free Body Diagrams depict the external forces that act on the mass(es).



Note that forces F_{k_2} and F_{c_2} appear in both free body diagrams.

Linear System Elements and "Getting The Signs Right"

The figure below shows a linear spring, a linear damper and their respective constitutive relations. Note that in this figure the forces shown are the external forces acting on the spring and damper which are equal and opposite to the forces acting on the masses, e.g. $f_k = -F_k$.



The magnitudes of the forces acting on the masses are given by

$$F_{k_1} = k_1 x_1 \qquad F_{k_2} = k_2(x_2 - x_1)$$
$$F_{c_1} = c_1 \dot{x}_1 \qquad F_{c_2} = c_2(\dot{x}_2 - \dot{x}_1)$$
$$F_{c_3} = c_3 \dot{x}_1 \qquad F_{c_4} = c_4 \dot{x}_2$$

Equations of Motion

Noting the directions on the free body diagrams, we can sum forces on m_1 and m_2,

$$\Sigma F_x = m_1 a_1 \rightarrow \qquad F_{k_2} + F_{c_2} - F_{k_1} - F_{c_1} - F_{c_3} = m_1 \ddot{x}_1$$
$$\Sigma F_x = m_2 a_2 \rightarrow \qquad F(t) - F_{k_2} - F_{c_2} - F_{c_4} = m_2 \ddot{x}_2$$

or

$$m_1 \rightarrow \qquad k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) - k_1 x_1 - c_1 \dot{x}_1 - c_3 \dot{x}_1 = m_1 \ddot{x}_1$$
$$m_2 \rightarrow \qquad F(t) - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - c_4 \dot{x}_2 = m_2 \ddot{x}_2$$

Rearranging,

$$m_1 \ddot{x}_1 + (c_1 + c_2 + c_3)\dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \qquad (1)$$

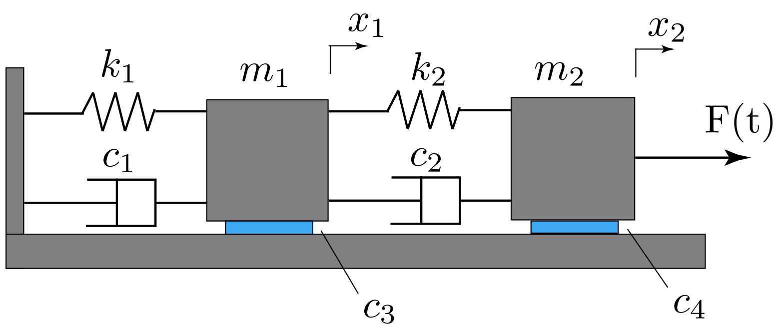
$$m_2 \ddot{x}_2 - c_2 \dot{x}_1 + (c_2 + c_4)\dot{x}_2 - k_2 x_1 + k_2 x_2 = F(t) \qquad (2)$$

In Matrix Notation

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 + c_3 & -c_2 \\ -c_2 & c_2 + c_4 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \quad (3)$$

or

$$\underline{M\ddot{x}} + \underline{C\dot{x}} + \underline{Kx} = 0$$



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