Rigid Body Dynamics

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October 21, 2007

1.0 Where are we in the course?

During the class of September 19th (about a month ago) I finished our coverage of kinematics of frames, systems of particles, and to a large extent, rigid bodies. We developed three key concepts. Do you remember what they were?

1. The concept of frames of reference, and derivatives with respect to frames. This is the key concept. The concepts below can be derived from this first statement, and are simply matters of convenience.
2. The magic formula for simplifying the taking of derivatives:

$$\frac{{}^A d\boldsymbol{r}}{dt} = \frac{{}^B d\boldsymbol{r}}{dt} + {}^A\omega^B \times \boldsymbol{r}\,.$$

3. The super-ultra magic formula which you can use when there is a particle moving with respect to a frame of reference which is itself moving:

$${}^A\boldsymbol{a}^S = {}^A\boldsymbol{a}^P + {}^B\boldsymbol{a}^S + {}^A\alpha^B \times \boldsymbol{r}^{PS} + {}^A\omega^B \times ({}^A\omega^B \times \boldsymbol{r}^{PS}) + 2\,{}^A\omega^B \times {}^B\boldsymbol{V}^S\,.$$

During the subsequent three lectures, I covered kinetics of a single particle (momentum, Newton's laws, work-energy principle, angular momentum) and collisions.

At that time, in terms of a roadmap of the course, we were as shown below. We said we would start examining the kinetics and the constitutive relationships of systems of particles and proceed to rigid bodies.

System	**Kinematics**	**Kinetics & Constitutive**
Particle	✓	✓
System of particles	✓	⌛
Rigid Bodies	✓	⌛
Lagrangian formulation		
Oscillations		

1.1 The Dumbbell Problem: Why Did we Do It?

During the lecture on the 10th of October, I analyzed the equations of motion of a dumbbell. Why did I do it? There were two reasons:

1. First, notice that I solved the dumbbell problem using nothing more than the kinematics that we had just derived, and Newton's Laws. The primary point I was making was that we actually have all the basic machinery to solve complex problems with multiple par-

ticles, like the dumbbell. It might be tedious, but it is there. Now we seek ways to simplify the analysis.

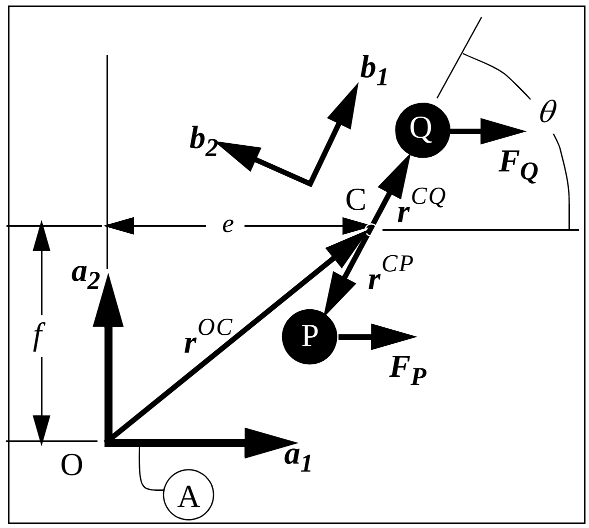
2. We solved the dumbbell problem the hard way: by formulating the equations of motion for the particles individually. Each particle was treated as if it had two degrees of freedom in 2D. We modeled the constraint that the particles couldn't move with respect to each other, and we explicitly put down the internal forces, which we also cancelled.[1] How do we simplify the analysis?

   Well, when we did the analysis, this is what we found that the equations of motion boiled down to:

$$\mathbf{F}_P + \mathbf{F}_Q = m\,{}^A\mathbf{a}^C \qquad \text{(EQ 1)}$$

   and

$$\mathbf{r}^{CQ} \times (\mathbf{F}_P - \mathbf{F}_Q) = (2ml^2)\,{}^A\alpha^B \qquad \text{(EQ 2)}$$



   In other words, for this rigid body:

   - the **total force** equals **mass times centroidal acceleration**;

   and

   - the **total torque** equals a new term called the **moment of inertia multiplied by angular acceleration**.

That's a total of 3 equations.

2.0 The Dynamics of Rigid Bodies

This is a fantastic discovery if we can generalize it. The consequences:

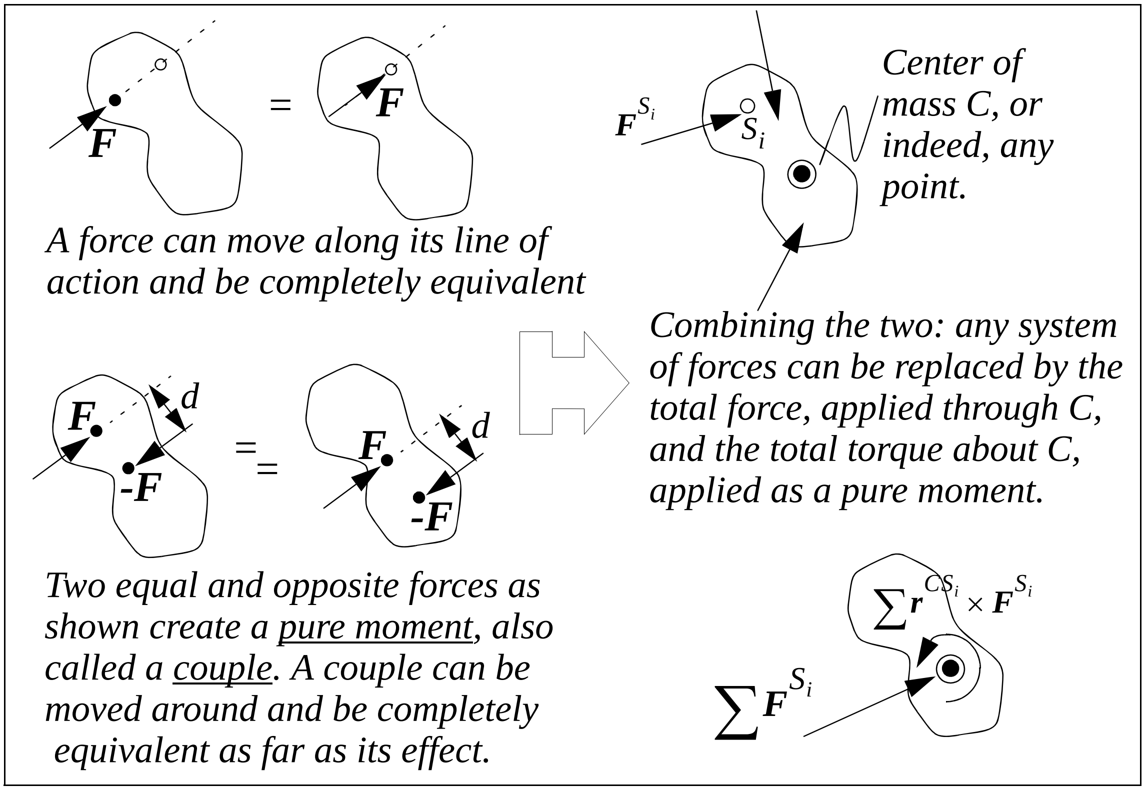
- In 2D, a rigid body has 3 degrees of freedom—two translation and one rotation. Guess what, the vector equations of the form above would give us 3 scalar equations—just what we need.
- In 3D, a rigid body has 6 degrees of freedom—three translation and three rotation. Guess what, vector equations of the type above would give us 6 scalar equations—just what we need.

Here's the great part: we can indeed generalize Equations 1 and 2. Let's recap where we ended up for rigid bodies. First, though, a small digression on force-torque equivalence.

1. When I was in college, I had a friend who misread the term "internal forces" and thought they were called "infernal forces." I thought it was a very apt Freudian slip.

2.1 Equivalence of Force-Torque Systems

A key principle in rigid-body dynamics, which everyone takes for granted is the principle of equivalence of different force-torque systems. The principles of equivalence are summarized below.



a) A force through a point can be replaced with the same force through any other point which is in the same line and have the exact same effect on the rigid body.

b) A perfect couple of forces (i.e., equal and opposite forces separated by some perpendicular distance) can be moved to any point on the rigid body and still have the same effect. This is called a pure moment or a couple.

c) Any system of forces can be replaced by an equivalent system of forces where:

- There is one force, equal to the total of all the forces, through the center of mass;
- And one pure moment, equal to the torque of all the forces about the center of mass.
- In fact you can do this with any point, not just the center of mass.

This cool principle reduces any complex set of forces on a rigid body to a canonical form consisting of one force (a vector in 2D) and a torque (effectively a scalar in 2D). OK, with that done, we now examine the equations of motion.

2.2 Equations for a Rigid Ensemble of Particles

The acceleration of the center of mass of a rigid body is related to the sum of forces in a very intuitive way:

$$\boldsymbol{F}^E = \frac{{}^A\mathrm{d}}{\mathrm{d}t} m\,{}^A\boldsymbol{v}^{C_E} = m\,{}^A\boldsymbol{a}^{C_E} \quad \text{(EQ 3)}$$

where $\boldsymbol{F}^E$ is the total force on the rigid body E and all the other terms are obvious by context. A, of course, is the inertial frame of reference. This gives us two equations in 2D (3 in 3D).

Furthermore, the torque about a point Q is related to angular momentum by:

$$\boldsymbol{T}^{E/Q} = \frac{{}^A\mathrm{d}}{\mathrm{d}t} {}^A\boldsymbol{H}^{E/Q} + {}^A\boldsymbol{v}^Q \times {}^A\boldsymbol{P}^E \quad \text{(EQ 4)}$$

where $\boldsymbol{T}^{E/Q}$ is the total torque of all the forces acting on rigid body E about the point Q; ${}^A\boldsymbol{H}^{E/Q}$ is the angular momentum of the rigid body E calculated about point Q and ${}^A\boldsymbol{P}^E$. That gives one more equation (3 more equations in 3D). For a free moving rigid body, therefore, we will get three differential equations for the three unknowns in 2D (and six equations for the six unknowns in 3D). Summary:

TABLE 1. Equations and unknowns in rigid body dynamics.

Dimensions	Unknowns (dof) related to motion	Equations from translation kinetics	Equations from rotation kinetics
2D	**3** (2 translation + 1 rotation)	**2** from Equation 3: $\boldsymbol{F}^E = m\,{}^A\boldsymbol{a}^E$	**1** (from Equation 4)
3D	**6** 3 translation + 3 rotation	**3** from Equation 3: $\boldsymbol{F}^E = m\,{}^A\boldsymbol{a}^E$	**3** (from equivalent of Equation 4)

This is all well and good, Sanjay, you might say. But how do we calculate the right-hand-side of Equation 4? Besides, what about that last term in Equation 4. Doesn't it stick out like a sore thumb? Indeed it does. Let's figure out how we get rid of it and put the equation to work.

2.3 Simplifying $\frac{{}^A\mathrm{d}}{\mathrm{d}t} {}^A\boldsymbol{H}^{E/Q} + {}^A\boldsymbol{v}^Q \times {}^A\boldsymbol{P}^E$

In class we took this opaque equation and tried to simplify it. First, we assume that Q We used kinematics to expand:

$$ {}^{A}\boldsymbol{H}^{E/Q} = \sum_{i=1}^{n} {}^{A}\boldsymbol{h}^{i/Q} = \sum_{i=1}^{n} \boldsymbol{r}^{Qi} \times m\, {}^{A}\boldsymbol{v}^{i} \qquad \text{(EQ 5)} $$

by breaking up the position vector $\boldsymbol{r}^{Qi}$ as $\boldsymbol{r}^{QC} + \boldsymbol{r}^{Ci}$ where C is the center of mass of the ensemble. We then computed the derivative of this expression and inserted back into the RHS of Equation 4 to arrived at the following after a great deal of calisthenics:

$$ \boldsymbol{T}^{E/Q} = \frac{{}^{A}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}^{OC} \times {}^{A}\boldsymbol{p}^{C}) + \frac{{}^{A}\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} m_i \boldsymbol{r}^{QC} \times [{}^{A}\omega^{B} \times \boldsymbol{r}^{Ci}]\right) + I^{E/Q}\, {}^{A}\alpha^{E} + {}^{A}\boldsymbol{v}^{Q} \times {}^{A}\boldsymbol{P}^{E} . \qquad \text{(EQ 6)} $$

Messy, huh? Who would have thought that the equation for $\boldsymbol{T}^{E/Q}$ isn't simply $I^{E/Q}\, {}^{A}\alpha^{E}$. There are some very specific conditions under which the mess in Equation 6 simplifies. The simplifications are below.

2.3.1 Simplification #1: About the Center of Mass

If the point Q is actually the center of mass then:

$$ \boldsymbol{T}^{E/C} = I^{E/C}\, {}^{A}\alpha^{E} . \qquad \text{(EQ 7)} $$

Simple, elegant, and always correct. *When in doubt, always use this equation about the center of mass.* Taking torques about the center of mass is always valid. The downside is that certain pesky external contact forces will show up as torques; this is slightly annoying but merely an algebraic nuisance, not a real problem.

2.3.2 Simplification #2: About the Instantaneous Center of Rotation

The instantaneous center of rotation (ICR) of a rigid body is specifically a point on the rigid body which is instantaneously at rest with respect to an (the) inertial frame. If the point Q is actually the (ICR) for the rigid body, then:

$$ \boldsymbol{T}^{E/(ICR)} = I^{E/(ICR)}\, {}^{A}\alpha^{E} . \qquad \text{(EQ 8)} $$

Also simple and elegant. The instantaneous center of rotation is often a hinge or, in rolling, the point of contact. Contact forces usually pass through this point, and so taking moments about this point is convenient because a lot of irrelevant forces will vanish. The downside is that the ICR might not always be a nice convenient point.

2.3.3 Simplification #3: My Grandmother Lives in Quito

If, for sentimental or technical reasons, you must take moments about some point Q which is neither he center of mass nor the ICR, then the formula is:

$$\boldsymbol{T}^{E/Q} = I^{E/C_E}\ {}^A\alpha^E + m\boldsymbol{r}^{QC_E} \times {}^A\boldsymbol{a}^C. \tag{EQ 9}$$

This general formula is useful when you feel that taking moments about this particular Q is advantageous because several unknown forces are eliminated from the equation, thereby simplifying your algebra.

Here are two cool observations.

- When Q is the center of mass, Equation 9 reduces seamlessly to Equation 7. Why? Because $\boldsymbol{r}^{QC_E} = 0$ in that case.
- When Q is the ICR, Equation 9 reduces seamlessly to Equation 8. Why? Because in that case, ${}^A\boldsymbol{a}^C = {}^A\alpha^C \times \boldsymbol{r}^{QC_E}$ and the second term becomes a restatement of the parallel axis theorem!

3.0 Energy: The Easy Free Equation

You saw for a single particle that you can always derive <u>one free equation</u> by invoking the work energy principle for rigid bodies. This is not extra — if you derive all possible equations from Newton's Laws, the energy equation is simply a restatement and will not add information. However, it is a freebie for the lazy amongst us. Lazy is fine in dynamics — a lot of what we have developed is convenience for efficiency. Everything could be done *ab initio* from Newton's Laws, after all, and we are deriving everything else just so that we can be more productive with our time.

The form of the work-energy relationship is simply: the work done by external forces equals the sum of the increases in kinetic energy and the potential energy.

3.1 Work Done by External Forces

The work done by a force $\boldsymbol{F}$ through the center of mass of a rigid body and a moment (torque) $\boldsymbol{T}$ about that rigid body is given by:

$$W_{1 \to 2} = \int_1^2 F \bullet d\boldsymbol{r} + \int_1^2 \boldsymbol{T} \bullet d\theta \tag{EQ 10}$$

where $d\boldsymbol{r}$ is the motion of the center of mass and $d\theta$ is the rotation of the rigid body (the infinitesimal rotation is a vector with a direction described by the screw rule.) $W_{1 \to 2}$ is the work done in a path traversed by a rigid body in from Point 1 to a Point 2.

- A force that is not through the center of mass will do work similarly, but the $d\boldsymbol{r}$ would be of that point. If there are many forces, the work contributions of all can be added up.

- The sum can be shown to be equivalent to work done by the total force through the center of mass and the work done by a pure moment. In other words, our equivalence reduction holds here.

3.2 Potential Energy

The potential energy of a rigid body depends on situation. If there are springs, then the springs can store energy. Under gravity, the formula is simple:

$$V_2 - V_1 = mg(\boldsymbol{r}^{\mathbf{1} \to \mathbf{2}} \bullet \boldsymbol{a}_{vertical}) \tag{EQ 11}$$

where $\boldsymbol{r}^{\mathbf{1} \to \mathbf{2}}$ is the vector from the beginning to the end of the path traversed by the center of mass of the rigid body, and where $\boldsymbol{a}_{vertical}$ is a unit vector pointing upwards!

3.3 Kinetic Energy

3.3.1 Kinetic Energy About the Center of Mass

We already know how to calculate the energy of a particle in translation. The kinetic energy of a rigid body has two contributions: one from translation and one from rotation. The translation term has a pleasing similarity to the energy of a particle. In fact it is equivalent to the energy of a single particle, located at the center of mass of the rigid body, and having the same mass as the rigid body:

$$T_{translation} = \frac{1}{2} m^E ({}^A v^{C_E})^2 \tag{EQ 12}$$

where A is an inertial frame of reference. The kinetic energy from rotation has a similar form (isn't that great?):

$$T_{rotation} = \frac{1}{2} I^{E/C_E} ({}^A \omega^E)^2. \tag{EQ 13}$$

The total energy is:

$$T = \frac{1}{2} m^E ({}^A v^{C_E})^2 + \frac{1}{2} I^{E/C_E} ({}^A \omega^E)^2. \tag{EQ 14}$$

Energy formulations have no pesky terms, no complications. The kinetic energy is the total energy stored in motion. That's it.

3.3.2 Kinetic Energy about the ICR

You can also calculate the total kinetic energy of a rigid body about the instantaneous center of rotation (ICR.) About the ICR, you don't need to account for the translational and rotational kinetic energies separately. Instead, the total kinetic energy is simply:

$$T = \frac{1}{2} I^{E/ICR} \left({}^{A}\omega^{E} \right)^{2}. \qquad \text{(EQ 15)}$$

3.4 The Work-Energy Principle

The total energy of a system, U, is $T + V$ where T and V are defined in the equations above. The work-energy principle simply relates the work done from state 1 to state 2 with the change in energy:

$$W_{1 \to 2} = U_2 - U_1 = (V_2 + T_2) - (V_1 + T_1), \qquad \text{(EQ 16)}$$

assuming that energy is not "leaked" anywhere.

4.0 Power

The power is simply the rate at which work is done on a system by a set of forces and is given by:

$$\boldsymbol{F}^{E} \bullet {}^{A}\boldsymbol{v}^{C_E} + \boldsymbol{T}^{E/C_E} \bullet {}^{A}\omega^{E}. \qquad \text{(EQ 17)}$$

The outcome is an equivalent rate of increase of the energy of the system, and is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}[T + V]. \qquad \text{(EQ 18)}$$

Again, if the system is conservative, and not energy is lost, then the two terms are equal.

5.0 How to Lay Out the Equations of Motion

There are two basic steps to writing the equations of motion:

1. Draw a **Free Body Diagram** (FBD).
   - Draw the object and all the forces on the object, showing there the act, along with the given dimensions.
   - Draw the inertial frame of reference, and all necessary intermediate frames of reference.
   - Label all the coordinates on the system: the unknown θ's, the l's and so on.
   - Pick a point about which you will take moments. Pick wisely depending on whether this point will simplify the problem or not. You will never go wrong if you pick the

center of mass. But you might go long. However, when in doubt, this is the best option.

2. Write down the equations of motion. Count the unknowns and Equations: you should be all squared away.
   - Write out any kinematic and constitutive equations you have. Calculate the appropriate moments of inertia.
   - Eliminate all the internal forces and superfluous coordinates, and come up with the number of equations you need for the number of dof's you have in terms of the remaining coordinates.
3. Oh, and solve the equations!! We will do that only a bit in this class. The bulk of the solving will occur in 2.004!!