Rigid Body Dynamics

Professor Sanjay Sarma

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1.0 Where are we in the course?

Thus far we have completed Kinematics and Kinetics of single particles, systems of particles and rigid bodies respectively. We are now well into the Lagrange portion of the class.

System	Kinematics	Kinetics & Constitutive
Particle	✓	✓
System of particles	✓	✓
Rigid Bodies	✓	✓
Lagrangian formulation	☞	
Oscillations	Next	

2.0 Generalized Coordinates

The generalized coordinates of a mechanical system are the minimal group of parameters which can completely and unambiguously define the configuration of that system. Some generalized coordinates are more “natural” than others, but there might be many ways to define them for any one system. The number of generalized coordinates equals the number of degrees of freedom of the system as long as the system is *holonomic*. We only study holonomic systems in this class.

Consider a system consisting of N rigid bodies in 2D space. Each rigid body has 3 degrees of freedom: two translational and one rotational. The N-body system has $3n$ degrees of freedom. Now let’s say that there are k kinematic constraints which can be expressed as algebraic equations. Then the system has $d = 3N - k$ degrees of freedom.

The term “holonomic” refers to the fact that the kinematic constraints must be expressible as algebraic equalities. Some kinematic constraints can only be expressed as inequalities or differential equations. Such systems are called non-holonomic constraints. We will not consider non-holonomic systems in this class— if you are interested in such systems, you can talk to me about them outside class.

3.0 Why Lagrange?

There are several reasons why the Lagrange Approach is important.

1. The Lagrange Approach automatically yields as many equations as there are degrees of freedom. It has the convenience of energy methods, but whereas energy conservation

only yields just one equation, which isn't enough for a multi-degree-of-freedom system, Lagrange yields as many equations as you need.

2. The Lagrange equations naturally use the generalized coordinates of the system. By contrast, Newton's Equations are essentially Cartesian. You end up having to convert everything into Cartesian components of acceleration and Cartesian components of forces to use Newton's Equation. Lagrange bypasses that conversion.

3. The Lagrange approach naturally eliminates non-contributing forces. You could do the same with the direct (Newtonian) approach, but your ability to minimize the number of variables depends very much on your skill; Lagrange takes care of it for you automatically because the generalized forces only include force components in *directions of admissible motion*.

4.0 The Lagrange Equations

For a d-dof (degree-of-freedom) system with generalized coordinates q_j's, it is possible to formulate the Lagrangian $L = T - V$ where T is the kinetic energy and V is the potential energy. The Lagrangian is a function of generalized coordinates q_j's and generalized velocities $\dot{q}_j$'s:

$$L = L(q_1, \ldots q_j \ldots q_d, \dot{q}_1, \ldots \dot{q}_j \ldots \dot{q}_d)\,.^1 \qquad \text{(EQ 1)}$$

where d is the number of degrees of freedom.

The Lagrange Equations are then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j\,, \qquad \text{(EQ 2)}$$

where Q_j's are the external generalized forces. Since j goes from 1 to d, Lagrange gives us d equations of motion.

But what are generalized forces? We derived them in class. Read on.

4.1 Generalized Forces

The generalized force Q_j is defined below:

1. There are some situations in which the Lagrangian is explicitly a function of time. Such systems are called *rheonomic* systems. We will not explore the implications in this course.

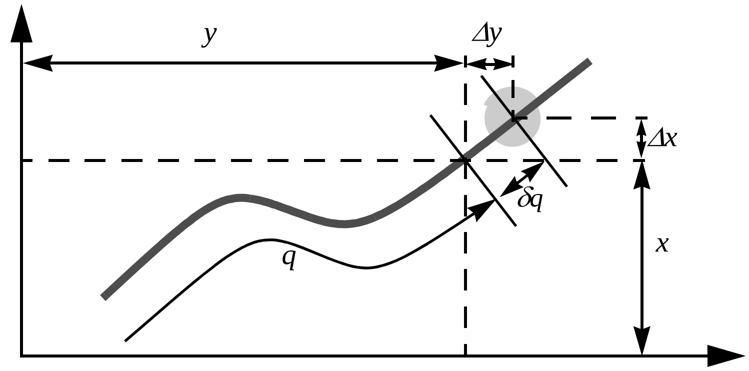

FIGURE 1. A bead on a wire

$$Q_j = \sum_{i=1}^{N} \boldsymbol{F}_i \bullet \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) \qquad \text{(EQ 3)}$$

where $\boldsymbol{F}_i$ is the force at point *i* and $\boldsymbol{r}_i$ is the position vector of point *i*. The index *j* corresponds to generalized coordinates.

4.2 The Intuition

So why does the Lagrange formulation work? The insight is simple. The Lagrange formulation only considers *admissible motions*.

4.2.1 The Problem with the Newtonian Approach

Consider a bead sliding without friction on a curved wire as shown in Figure 2. Clearly the bead can only move along the wire, which can be approximated locally as a direction tangential to the wire. Now, the Cartesian coordinates of the bead would be *x* and *y*. However, these coordinates are redundant. We can only eliminate the redundancy by introducing a geometric constrain between *x* and *y* of the form $Constraint(x, y) = 0$.[1] For example, if the wire is in the form of a circle of radius *R*, the constraint will be $x^2 + y^2 - R^2 = 0$. No combination of Δx and Δy is legal if it does not satisfy $x^2 + y^2 - R^2 = 0$.

In the direct, or Newtonian approach, we waste a lot of time considering *x* and *y* motions as if the bead could get to any *x* and *y* (which it can't), postulating reaction forces (which are actually irrelevant) and then solving for these reaction forces and motions such that the Δx and Δy satisfy the kinematic constraint (which is a waste of time). The problem, as

1. We will assume that this is an algebraic. If it is an inequality constraint or an unintegrable differential equation, we need more machinery which we will not cover in this course.

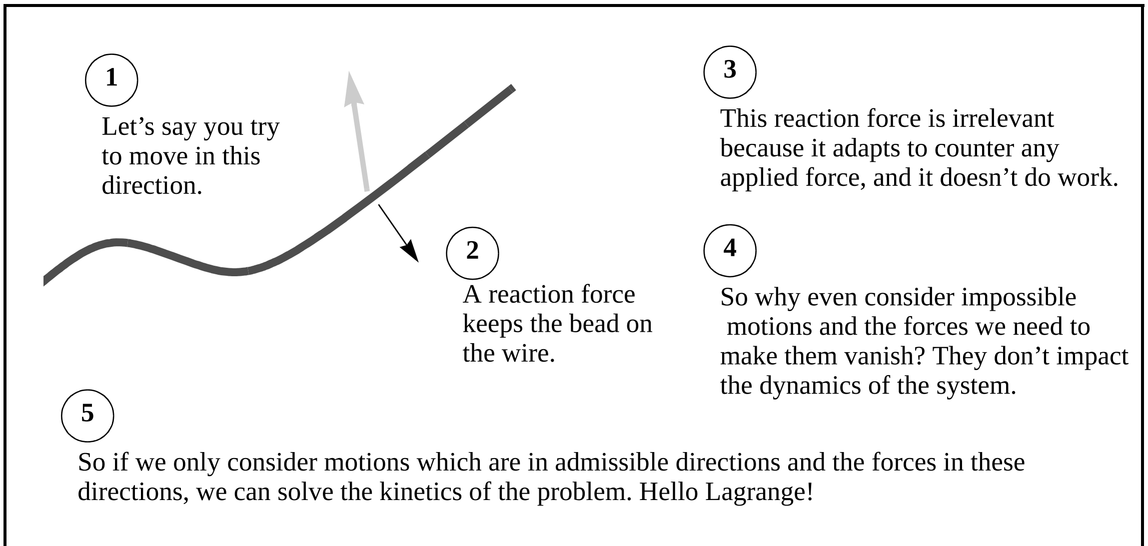

FIGURE 2. Admissible motions and the non-contributing forces that enforce them

shown in Figure 2, is that we do everything explicitly and in the process, we end up solving for a number of extra variables like reaction forces and inadmissible motions which end up being irrelevant to the actual dynamics of the system. Essentially, pushing at an immovable object causes to motion.

4.2.2 Admissible Motions

Here's the rub. The use of a good set of generalized coordinates eliminates this problem because *generalized coordinates implicitly capture admissible motions*. For example, if our wire is in the shape of a circular loop, an appropriate generalized coordinate is the angle of bead on the wire loop as shown in Figure 3 (a). If our wire were a cosine shape, it would look like Figure 3 (b) (I will concentrate on the circle in these notes, and leave it to you to work the math out for the sinusoid.) Now, consider the position vector $\boldsymbol{r}$ written as a function of q for the circular loop:

$$\boldsymbol{r}(q) = (R\cos q)\boldsymbol{a}_1 + (R\sin q)\boldsymbol{a}_2 .$$

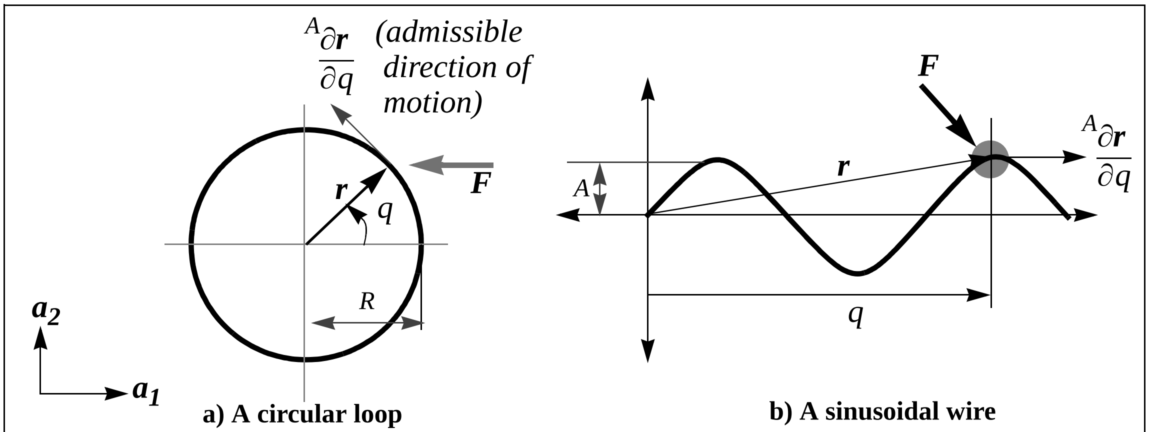

FIGURE 3. Different wire shapes and relevant generalized coordinates

Now consider the expression $\frac{{}^A\partial \boldsymbol{r}}{\partial q}$. (We were sloppy about specifying the frame for the derivative in the past, and we will omit it in the future under the assumption that when not stated, the frame of reference for a derivative is the inertial frame A.) Let's compute this expression:

$$\frac{\partial \boldsymbol{r}}{\partial q} = -(R\sin q)\boldsymbol{a}_1 + (R\cos q)\boldsymbol{a}_2 .$$

Guess what, this vector is tangential to the circle and <u>instantaneously captures the admissible motion of the bead</u>. A small variation of q, δq, results in a $\delta \boldsymbol{r}$ given by:

$$\delta r = \frac{\partial \boldsymbol{r}}{\partial q}\delta q . \qquad \text{(EQ 4)}$$

$\delta \boldsymbol{r}$ is an admissible motion for the bead. *It captures the kinematic constraint.* In general, in a *d*-degree-of-freedom system with generalized coordinates $q_1, \ldots, q_d$, the admissible motions at a point *i* with position vector $\boldsymbol{r_i}$ are given by:

$$\delta \boldsymbol{r_i} = \sum_{i=1}^{d} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j . \qquad \text{(EQ 5)}$$

Note that the symbol δ in front of a variable emphasizes that the motion is an implicitly admissible motion. The *d*-dimensional version is actually a d-dimensional tangent space just like in a 1-dof case.

4.2.3 $\boldsymbol{f} = m\boldsymbol{a}$ Written as Components in Admissible Directions

If you applied a force $\boldsymbol{F}$ on the bead shown in Figure 1, the only component which is relevant, assuming the wire is rigid, is the component of the force along the admissible direction. For the bead, this is given by $\frac{\partial \boldsymbol{r}}{\partial q}$. So the only force component we need to worry about is:

$$\boldsymbol{F} \bullet \frac{\partial \boldsymbol{r}}{\partial q} . \qquad \text{(EQ 6)}$$

All other forces are perpendicular to the motion and don't do any work! Of course, $\frac{\partial \boldsymbol{r}}{\partial q}$ isn't a unit vector, and its dimensions are those of a length, but don't worry about that for a moment. What we have just derived is the generalized force for a 1-dof system.

Newton's Law says $\boldsymbol{F} = m\boldsymbol{a}$. We have just accounted for the LHS along the admissible direction. Similarly, the only acceleration component we need to worry about is the one in an admissible direction, and the RHS of Newton's Equation of motion can be written as:

$$m\ddot{\boldsymbol{r}} \bullet \frac{\partial \boldsymbol{r}}{\partial q} \qquad \text{(EQ 7)}$$

when we recognize that $\boldsymbol{a} = \ddot{\boldsymbol{r}}$.

So taking the components of Newton's Laws in the admissible direction only, we get:

8 $$\boldsymbol{F} \bullet \frac{\partial \boldsymbol{r}}{\partial q} = m\ddot{\boldsymbol{r}} \bullet \frac{\partial \boldsymbol{r}}{\partial q}. \qquad \text{(EQ)}$$

or, looking at work, we get:

9 $$\boldsymbol{F} \bullet \frac{\partial \boldsymbol{r}}{\partial q}\delta q = m\ddot{\boldsymbol{r}} \bullet \frac{\partial \boldsymbol{r}}{\partial q}\delta q. \qquad \text{(EQ)}$$

Look familiar? This is how we started our derivation of Lagrange's Equations. This leads to the 1-dof Lagrangian Equation. The LHS is Q, the generalized force. Essentially, the RHS of Equation 8 reduces to:

$$Q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\ . \qquad \text{(EQ 10)}$$

Look up the derivation from class to see why. You can extend this reasoning to multi-dof systems and get the general Lagrangian Equation:

1 $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j. \qquad \text{(EQ 1)}$$

4.2.4 Generalized Forces Again

So the key matter regarding generalized forces is this:

- Forces of constraint which do not do work can be ignored because they will always be perpendicular to admissible directions. Examples include the internal forces in a rigid body, the forces of reaction in friction-less sliding, and so on.
- Forces which derive from a potential function like gravity or a spring can be considered in potential energy, V. They too can be ignored when computing generalized forces.
- Internal forces in rigid bodies do not contribute.
- Forces in pure rolling don't contribute.

- Forces which are none of the above need to be called out and used in Formula 3. We will call such forces contributing forces. Examples include dissipative forces from dashpots, externally applied forces and so on. You can't go wrong including a force in this category instead of one of those above because they will vanish or be accounted for appropriately here.

5.0 Using Lagrange's Equations

The steps in computing the equations of motion using Lagrange's method are below.

Start with the LHS of Equation 11:

1. Identify the generalized coordinates. Make sure that you have just as many as there are degrees-of-freedom.
2. Compute the kinetic energy T as a function of q_j 's and $\dot{q}_j$'s.
3. Compute the potential energy V as a function of q_j 's and $\dot{q}_j$'s. Clearly mark out the forces which you will call out as potential and forces which you will call out as external
4. Compute $L = T - V$, which will obviously be a function of q_j 's and $\dot{q}_j$'s.
5. Compute $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$ and $\frac{\partial L}{\partial q_j}$ and you have the LHS for each j.

Now the RHS of Equation 11:

1. Identify all contributing forces.
2. Number them as $i = 1, 2, \ldots n$. Call the forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots \boldsymbol{F}_n$.
3. Identify the precise points where the forces are applied on the system, and identify the position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ respectively for all these points. Each $\boldsymbol{r}_j$ must be a function of q_j 's.
4. For each j, compute the generalized force using Equation 3: $Q_j = \sum_{i=1}^{n} \boldsymbol{F}_i \bullet \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)$.

Now equate the LHS to RHS for each *j*.

---Done---