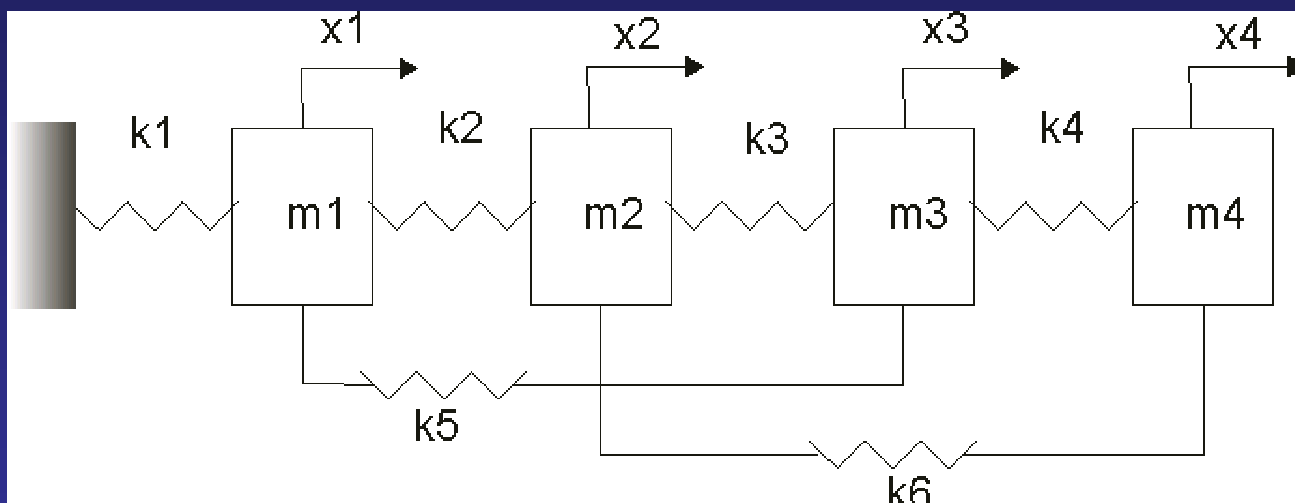
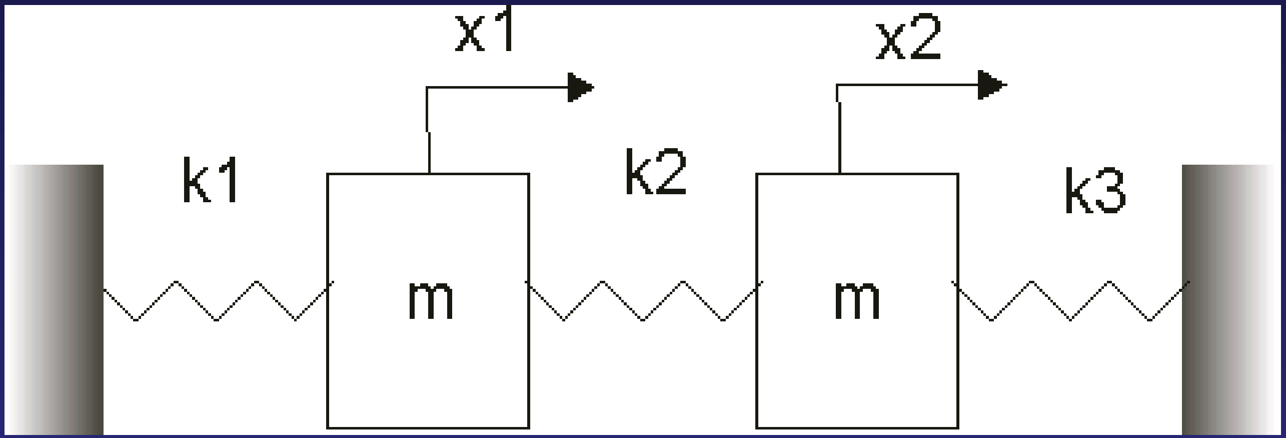
MATLAB Programming –
Eigenvalue Problems and Mechanical Vibration



$$A \cdot x = \lambda x \quad (A - \lambda I) \cdot x = 0$$

A Coupled Mass Vibration Problem



EOM:

$$m\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = 0$$
$$m\ddot{x}_2 + k_1 x_2 + k_2(x_2 - x_1) = 0$$

Vibration Solutions – harmonic response

Trial solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cos(\omega t + \varphi)$$

Matrix representation of EOM:

$$\begin{bmatrix} -\omega^2 + (k_1 + k_2)/m & -k_2/m \\ -k_2/m & -\omega^2 + (k_1 + k_2)/m \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

Vibrational Frequencies and Mode Shapes

Characteristic Equation (Determinant = 0):

$$k_1 + k_2 - m\omega^2 = \pm k_2$$

$$\omega_1^2 = \frac{k_1}{m} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_1 = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\omega_2^2 = \frac{k_1 + 2k_2}{m} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_2 = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Vibrations as a general class of "Eigenvalue Problems"

Recast EOM:

$$\begin{bmatrix} -\omega^2 + (k_1 + k_2)/m & -k_2/m \\ -k_2/m & -\omega^2 + (k_1 + k_2)/m \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

As:

$$\begin{bmatrix} (k_1 + k_2)/m & -k_2/m \\ -k_2/m & (k_1 + k_2)/m \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} (k_1 + k_2)/m & -k_2/m \\ -k_2/m & (k_1 + k_2)/m \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$A \cdot x = \lambda I \cdot x$$

$$(A - \lambda I) \cdot x = 0$$

Eigenvalue equation, Eigenvalues, Eigenvectors

Eigenvalue equation:

$$A \cdot x = \lambda x \quad (A - \lambda I) \cdot x = 0$$

Eigenvalues (angular frequencies of the vibration):

$$\lambda = \omega^2$$

Eigenvectors (mode shape of the vibration):

$$x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving Eigenvalue Problem in MATLAB

Look at the problem numerically:

$$m = 1kg \quad k_1 = 1N/m \quad k_2 = 2N/m$$

Simple m-file:

```
m=1;
k1=1;
k2=2;
A=[(k1+k2)/m -k2/m; -k2/m (k1+k2)/m]
[X,L]=eig(A);
X
L
```

MATLAB output of simple vibration problem

eigenvector 1 eigenvector 2

X =

-0.7071	-0.7071
-0.7071	0.7071

L =

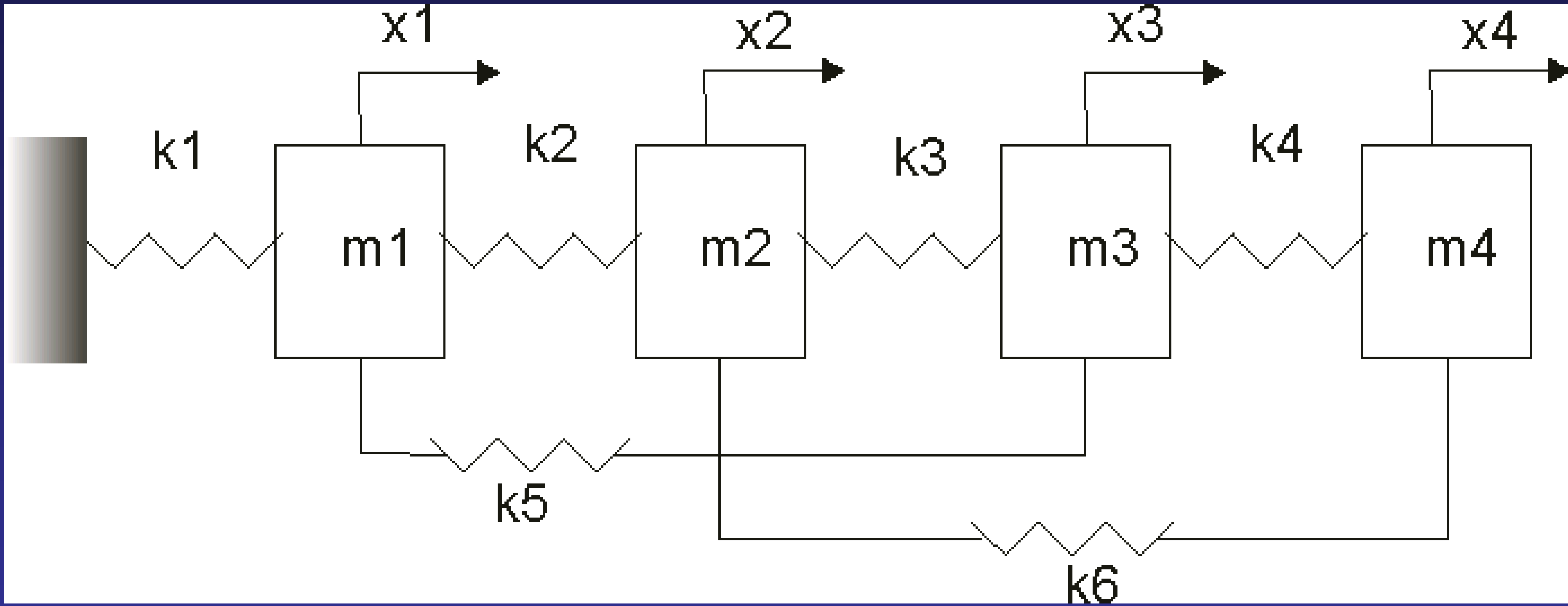
eigenvalue 1

eigenvalue 2

1.0000	0
0	5.0000

Ok, we get the same results as solving the characteristics equation… so what is the big deal?

A more complex vibrations problem



EOM for a more complex problem

EOM can be gotten from free body diagrams of each mass

$$m_1\ddot{x}_1 + (k_1 + k_2 + k_5)x_1 - k_2 x_2 - k_5 x_3 = 0$$

$$m_2\ddot{x}_2 + (k_2 + k_3 + k_6)x_2 - k_2 x_1 - k_3 x_3 - k_6 x_4 = 0$$

$$m_3\ddot{x}_3 + (k_3 + k_4 + k_5)x_3 - k_5 x_1 - k_3 x_2 - k_4 x_4 = 0$$

$$m_4\ddot{x}_4 + (k_4 + k_6)x_4 - k_6 x_2 - k_4 x_3 = 0$$

Characteristic equation of more complex problem:

$$\begin{bmatrix} -\omega^2 - \frac{k_1+k_2+k_5}{m_1} & \frac{k_2}{m_1} & \frac{k_5}{m_1} & 0 \\ \frac{k_2}{m_2} & -\omega^2 - \frac{k_2+k_3+k_6}{m_2} & \frac{k_3}{m_2} & \frac{k_6}{m_2} \\ \frac{k_5}{m_3} & \frac{k_3}{m_3} & -\omega^2 - \frac{k_2+k_4+k_5}{m_3} & \frac{k_4}{m_3} \\ 0 & \frac{k_6}{m_4} & \frac{k_4}{m_4} & -\omega^2 - \frac{k_4+k_6}{m_4} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

Solving this and find roots is no longer so simple!
It is now an eighth order polynomial….

Look at more complex vibration as eigenvalue problem

$$\begin{bmatrix} \frac{k_1+k_2+k_5}{m_1} & -\frac{k_2}{m_1} & -\frac{k_5}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3+k_6}{m_2} & -\frac{k_3}{m_2} & -\frac{k_6}{m_2} \\ -\frac{k_5}{m_3} & -\frac{k_3}{m_3} & \frac{k_2+k_4+k_5}{m_3} & -\frac{k_4}{m_3} \\ 0 & -\frac{k_6}{m_4} & -\frac{k_4}{m_4} & \frac{k_4+k_6}{m_4} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \omega^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Solving more complex problem by MATLAB

Pick some numerical values:

$$m_1 = 1kg \quad m_2 = 2kg \quad m_3 = 3kg \quad m_4 = 4kg$$

$$k_1 = 1N/m \quad k_2 = 2N/m \quad k_3 = 3N/m$$

$$k_4 = 4N/m \quad k_5 = 5N/m \quad k_6 = 6N/m$$

Solving complex vibration problem by MATLAB

Create another m-file:

```
m1=1;m2=2;m3=3;m4=4;
k1=1;k2=2;k3=3;k4=4;k5=5;k6=6;
A=[(k1+k2+k5)/m1 -k2/m1 -k5/m1 0;
-k2/m2 (k2+k3+k6)/m2 -k3/m2 -k6/m2;
-k5/m3 -k3/m3 (k3+k4+k5)/m3 -k4/m3;
0 -k6/m4 -k4/m4 (k4+k6)/m4]
[X,L]=eig(A);
X
L
```

Characteristic matrix of the eigenvalue problem

A =

8.0000	-2.0000	-5.0000	0
-1.0000	5.5000	-1.5000	-3.0000
-1.6667	-1.0000	4.0000	-1.3333
0	-1.5000	-1.0000	2.5000

Frequencies and mode shapes of complex problem

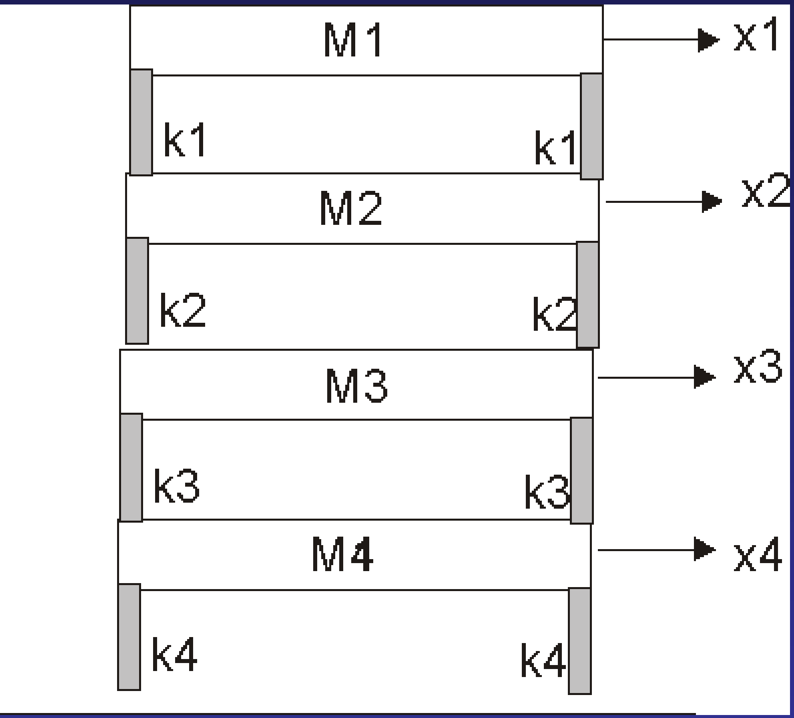
X =

0.4483	0.9456	0.6229	-0.1650
0.5156	-0.1826	-0.0411	-0.9006
0.5031	-0.2591	0.5788	0.3164
0.5292	0.0735	-0.5246	0.2481

L =

0.0878	0	0	0
0	9.7562	0	0
0	0	3.4858	0
0	0	0	6.6702

Multiple element vibration problems – finite element simulation



Consider the vibrational modes of a four stories building. The mass are assumed to be concentrated at the floors. The walls constitutes springs. This can be models as a 1-D system.

Inspired by Aladdin web site

Write equation of motion for the four floors

$$m_1\ddot{x}_1 = 2k_1(x_2 - x_1)$$

$$m_2\ddot{x}_2 = 2k_1(x_1 - x_2) + 2k_2(x_3 - x_2)$$

$$m_3\ddot{x}_3 = 2k_2(x_2 - x_3) + 2k_3(x_4 - x_3)$$

$$m_4\ddot{x}_4 = 2k_3(x_3 - x_4) - 2k_4x_4$$

Write down the Mass and Stiffness Matrix

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}$$

$$K = \begin{bmatrix} 2k_1 & -2k_1 & 0 & 0 \\ -2k_1 & (2k_1 + 2k_2) & -2k_2 & 0 \\ 0 & -2k_2 & (2k_2 + 2k_3) & -2k_3 \\ 0 & 0 & -2k_3 & (2k_3 + 2k_4) \end{bmatrix}$$

Using MATLAB Eig with Mass & Stiffness Matrix Directly

$$KV = MVD$$

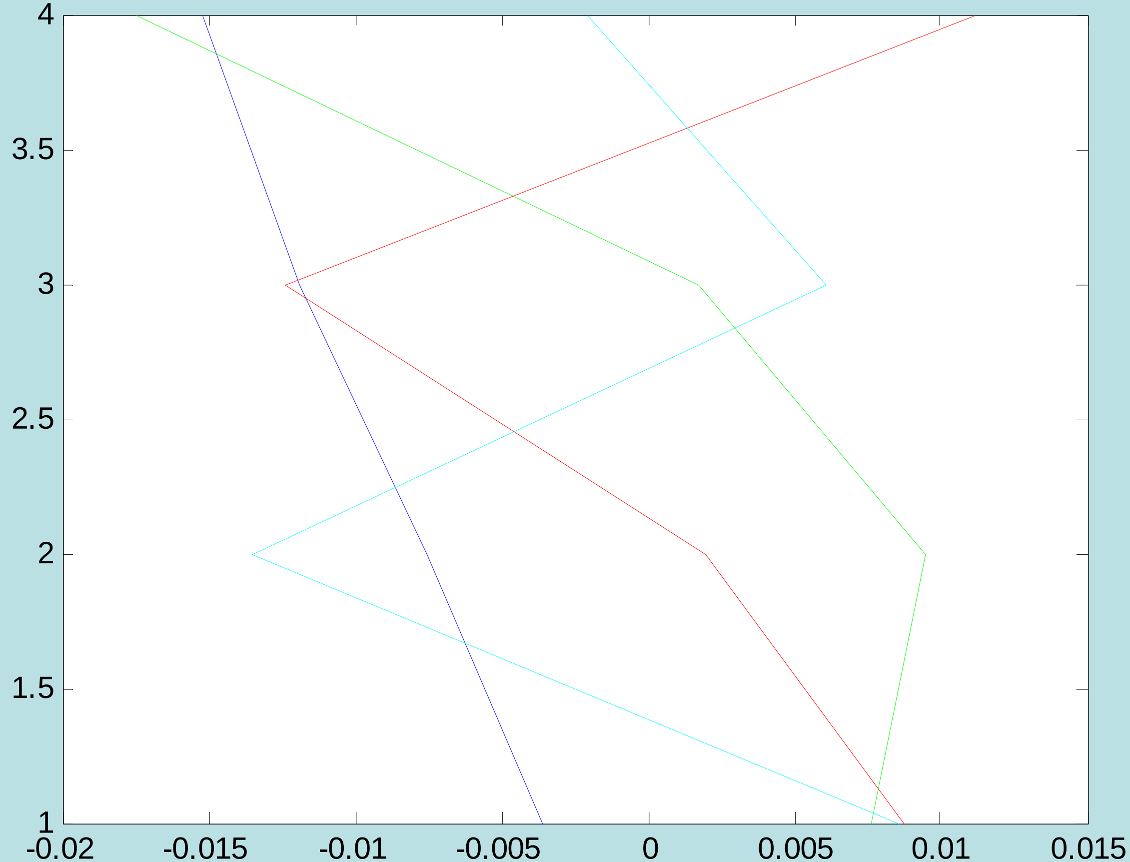
Eig can also operate on the eigenvalue equation
In this form where:
K is the stiffness matrix, V is the matrix containing
All the eigenvectors, M is the mass matrix, and
D is a diagonal matrix containing the eigenvalues

```
[V,D]=eig(K,M)
```

Mode shape of building oscillation

$$m_1 = 1500 \quad m_2 = 3000 \quad m_3 = 3000 \quad m_4 = 4500$$

$$k_1 = 400 \quad k_2 = 800 \quad k_3 = 1200 \quad k_4 = 1600$$

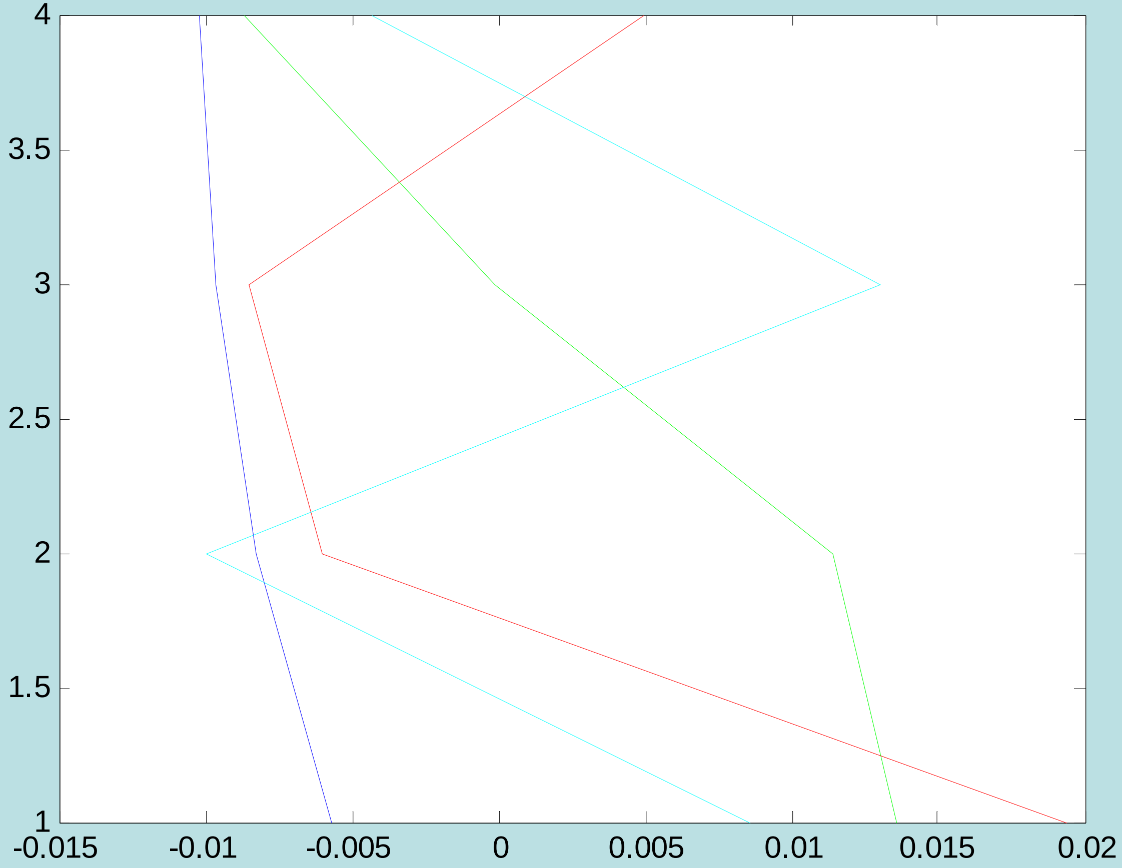


$$\omega_1^2 = 0.12 \quad \omega_2^2 = 0.59 \quad \omega_3^2 = 1.13 \quad \omega_4^2 = 2.08$$

Mode shape of building oscillation 2

$$m_1 = 1500 \quad m_2 = 3000 \quad m_3 = 3000 \quad m_4 = 4500$$

$$k_1 = 400 \quad k_2 = 800 \quad k_3 = 1200 \quad k_4 = 1600$$



$$\omega_1^2 = 0.042 \quad \omega_2^2 = 0.70 \quad \omega_3^2 = 1.93 \quad \omega_4^2 = 2.83$$