1.053J/2.003J Dynamics and Control I
Fall 2006

Final Exam
20th December, 2006

Important Notes:

1. You are allowed to use three letter-size sheets (two-sides) of notes.
2. There are five problems on the exam for a total of 100 points. The points awarded for each problem are indicated at the top.
3. You have 180 minutes.

Problem 1: Non-uniform disc (20 points)

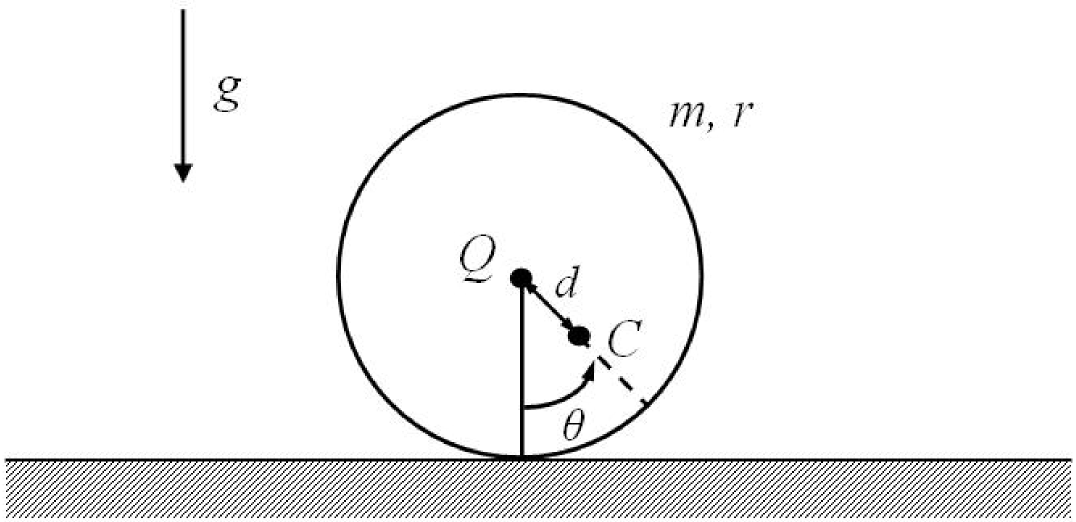

Figure 1

A non-uniform disc of mass *m* and radius *r* has its center of mass *C* at a distance *d* from the center of the disc *Q*, as shown in figure 1. The disc rolls on the horizontal surface without slippage.

Initially, the disc is oriented such that *C* lies vertically below *Q*, after which it is perturbed and starts rolling. Let θ be the instantaneous angle that the segment *QC* makes with the vertical direction, measured in the anti-clockwise direction as shown in the figure, and let *I* be the moment of inertia of the disc about point *C*. Note that gravity acts.

a. In terms of θ and its derivatives $\dot{\theta}$ and $\ddot{\theta}$, find the acceleration of *C* with respect to a frame that is fixed to the ground.

Now use the Newtonian method (linear and angular momentum equations) to derive the equation of motion for the system as follows.

b. Can you use the torque equation about the instantaneous center of rotation (ICR)? Explain.

c. Using the torque equation about the center of mass of the disc and any other relevant equations, show that the equation of motion is:

$$\left[I + m\left(r^2 + d^2 - 2rd\cos\theta\right)\right]\ddot{\theta} + mrd\,\dot{\theta}^2\sin\theta + mgd\sin\theta = 0$$

Problem 2: Non-uniform disc: equilibria and stability (15 points)

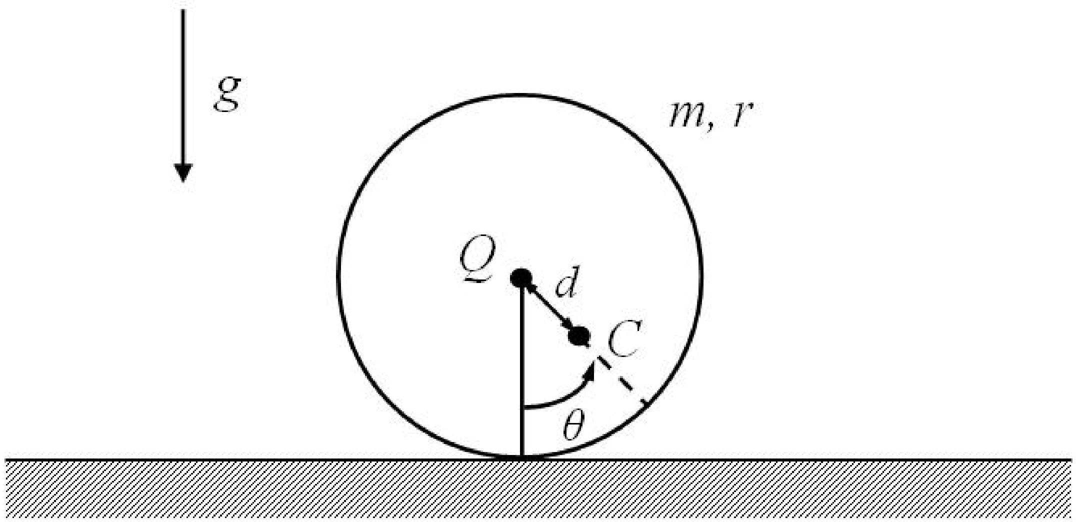

Figure 2

Consider the motion of the non-uniform disc studied in problem 1. You may use the following equation to describe it:

$$\left[I + m\left(r^2 + d^2 - 2rd\cos\theta\right)\right]\ddot{\theta} + mrd\,\dot{\theta}^2 \sin\theta + mgd\,\sin\theta = 0$$

a. For the above system, find the two equilibrium points greater than or equal to 0 and less than 2π.

b. Linearize the equation of motion about <u>both</u> the equilibrium points and assess the stability of each. You may ignore any quadratic or higher order terms that arise in terms of the perturbation variable or its derivatives. For the stable equilibrium, determine the natural frequency of oscillation.

Problem 3: A rolling disc between parallel rods (20 points)

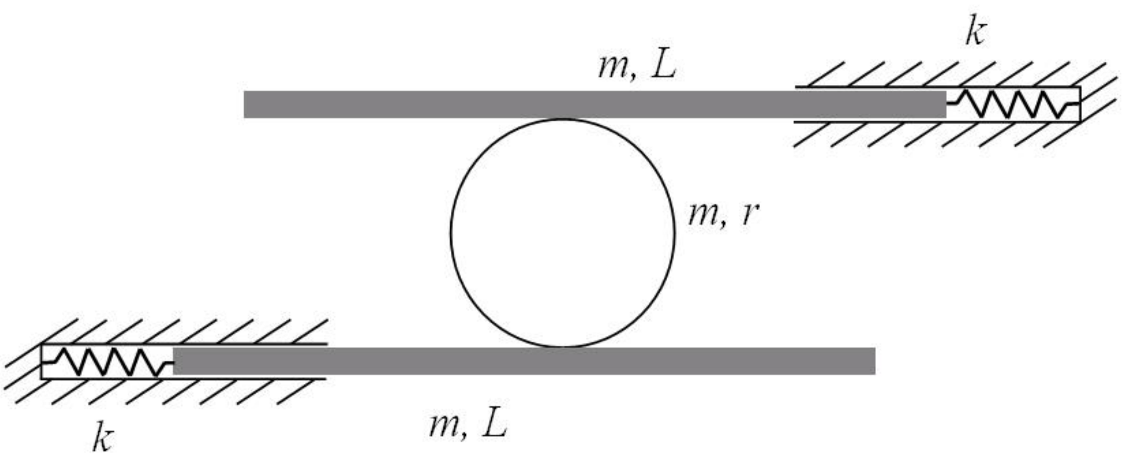

Figure 3

A disc of mass *m* and radius *r* is sandwiched between two slender rods, each of mass *m* and length *L*. Each rod can slide in and out of a horizontal groove without friction, as shown in figure 3. The system starts at rest in a position in which both the springs are relaxed. The disc *rolls without slippage* with respect to both rods. Each rod is connected to a massless spring with spring constant *k*, and the other end of the spring is attached to the inertial frame. The system starts moving when one of the rods is pulled out and released.

Ignore gravity and derive the equation(s) of motion for this system.

Hint: This system has two degrees of freedom. If you know the displacements of the rods then the displacement and rotation of the disc is uniquely determined. We recommend that you write this kinematic constraint out right at the outset.

Problem 4: Composite system (25 points)

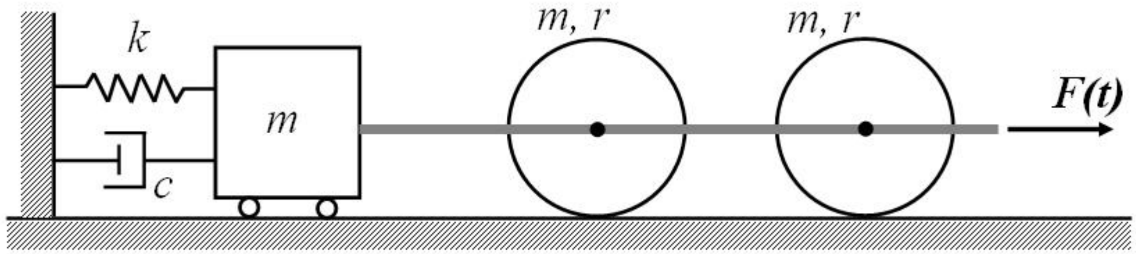

Figure 4

The system shown in figure 4 consists of four rigid-bodies as described below.

The first block rolls on small, massless wheels and is connected to a fixed wall using a massless spring with spring constant k and a massless dashpot with damping coefficient c. This block is connected to a massless, rigid rod with two bearings. At each bearing, a disc of mass m and radius r is mounted. The rod is horizontal. Each disc rolls on the horizontal surface without slippage. A horizontal force ***F(t)*** is applied to the free end of the rod.

a. Derive the equation(s) of motion for the system.
b. Reduce the equation(s) to an equivalent system consisting of a mass, a spring and a dashpot and an external force.
c. What is the natural frequency of this system? Determine the conditions under which the homogeneous solution to system is underdamped, critically damped and overdamped.

Now assume that $\boldsymbol{F(t)} = F_0 \cos(\Omega t)$.

d. What is the time response of the system under this forcing in the underdamped case?
e. If the damping coefficient is zero, for what value of Ω do you expect to see resonance?

Problem 5: A rolling disc on a see-saw (20 points)

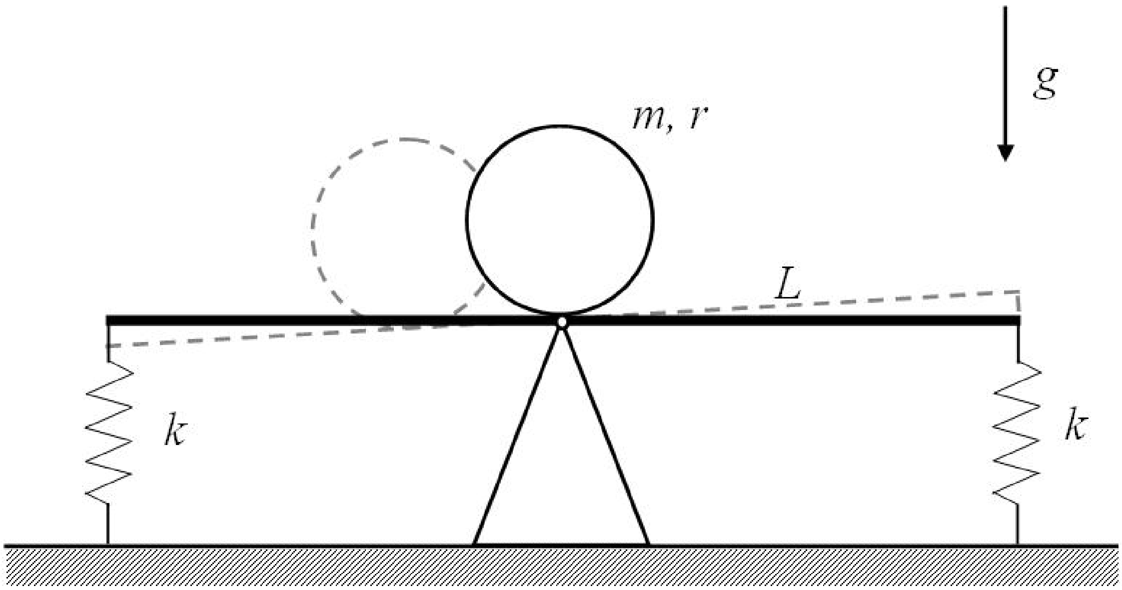

Figure 5

A rigid, massless rod of length *L* is free to rotate about a pivot through its center. Both ends of the rod are attached to two identical, massless springs with spring constant *k*. The other ends of the springs are fixed to the ground. Initially, the rod is horizontal and both the springs are un-stretched. A disc of mass *m* and radius *r* is placed at the center of the rod. The disc rolls on the rod without slippage. (The dashed lines in figure 5 show a possible alternative configuration of the system)

Choose appropriate generalized coordinate(s) for this system and derive the equation(s) of motion using a method of your choice. Note that gravity acts.