2.002 MECHANICS & MATERIALS II

STRESS-STRAIN RELATIONS

Introduction To Linear Elasticity

- Elastic materials have a reference shape to which they will return if the forces applied to them are removed (provided the forces are not too large).

- An elastic material is one in which the stress arises in response to the change in shape, that is the strain ϵ, that the body has undergone from its reference configuration.

- The stress is independent of the past history of strain, as well as the rate at which the strain is changing with time.

- The behavior of an **elastic material** under isothermal conditions is desribed by a constitutive equation of the form

$$\sigma_{ij} = \widehat{\sigma}_{ij}(\epsilon_{kl}), \quad \boldsymbol{\sigma} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}).$$

- For a **linear elastic material** the symmetric tensor valued function $\widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon})$ is linear in its argument:

$$\widehat{\sigma}_{ij}(\epsilon_{kl}) = \sum_{k,l} \mathcal{C}_{ijkl}\epsilon_{kl}, \quad \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}) = \boldsymbol{\mathcal{C}}\left[\boldsymbol{\epsilon}\right].$$

  The stress components are linear functions of the infinitesimal strain components. The fourth-order tensor $\boldsymbol{\mathcal{C}}$ which linearly maps the second order tensor $\boldsymbol{\epsilon}$ to the second order tensor $\boldsymbol{\sigma}$, is the called the **stiffness tensor**. The $3^4 = 81$ constants $\mathcal{C}_{ijkl}$ are called the **elastic moduli**.

- An elastic material **does not dissipate energy**.

- There exist a scalar valued function of the strain ϵ,

$$W(\epsilon)$$

  called the **strain energy density per unit reference volume**, such that the stress σ_{ij} is the derivative of $W(\epsilon)$; that is

$$\sigma_{ij} = \frac{\partial W(\epsilon)}{\partial \epsilon_{ij}}.$$

Expanding the strain energy function $W(\epsilon)$ about the undeformed state, $\epsilon = \mathbf{o}$, we have

$$W(\epsilon) = W(\mathbf{o}) + \sum_{i,j} \frac{\partial W(\mathbf{o})}{\partial \epsilon_{ij}} \epsilon_{ij} + \frac{1}{2} \sum_{i,j,k,l} \frac{\partial^2 W(\mathbf{o})}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \epsilon_{ij} \epsilon_{kl} + \cdots$$

$W(\mathbf{o}) = 0$ No strain energy at zero strain

$\sigma_{ij,\text{residual}} = \dfrac{\partial W(\mathbf{o})}{\partial \epsilon_{ij}}$ Residual stress at zero strain, neglect

$\mathcal{C}_{ijkl} \equiv \dfrac{\partial^2 W(\mathbf{o})}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$ Elastic moduli – constants; $3^4 = 81$!

Thus, for a **linear elastic material**,the strain energy density function W is quadratic in ϵ:

$$W(\epsilon) = \frac{1}{2} \sum_{i,j,k,l} \mathcal{C}_{ijkl} \epsilon_{ij} \epsilon_{kl}.$$

$$W(\epsilon) = \frac{1}{2} \sum_{i,j,k,l} \mathcal{C}_{ijkl} \epsilon_{ij} \epsilon_{kl}$$

$$\frac{\partial^2 W(\mathbf{o})}{\partial \epsilon_{ij} \partial \epsilon_{kl}} = \frac{\partial^2 W(\mathbf{o})}{\partial \epsilon_{kl} \partial \epsilon_{ij}} \quad \Longrightarrow \quad \mathcal{C}_{ijkl} = \mathcal{C}_{klij}$$

$$\epsilon_{ij} = \epsilon_{ji} \Longrightarrow \quad \mathcal{C}_{ijkl} = \mathcal{C}_{jikl}$$

$$\epsilon_{kl} = \epsilon_{lk} \Longrightarrow \quad \mathcal{C}_{ijkl} = \mathcal{C}_{ijlk}$$

these symmetries reduce the number of independent elastic constants to 21.

Now,

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \frac{1}{2}\frac{\partial}{\partial \epsilon_{ij}}\left\{\sum_{p,q,r,s} \mathcal{C}_{pqrs}\epsilon_{pq}\epsilon_{rs}\right\},$$
$$= \frac{1}{2}\sum_{p,q,r,s} \mathcal{C}_{pqrs}\left(\delta_{ip}\delta_{jq}\epsilon_{rs} + \delta_{ir}\delta_{js}\epsilon_{pq}\right),$$
$$= \frac{1}{2}\sum_{r,s}\left(\mathcal{C}_{ijrs}\epsilon_{rs} + \mathcal{C}_{pqij}\epsilon_{pq}\right) = \sum_{r,s}\mathcal{C}_{ijrs}\epsilon_{rs}.$$

That is,

$$\boxed{\sigma_{ij} = \sum_{k,l}\mathcal{C}_{ijkl}\,\epsilon_{kl},}$$

which, as noted before, is the constitutive equation for a linear elastic solid. The stress components are linear functions of the infinitesimal strain components.

Constitutive equation for a linear elastic solid:

$$\boxed{\sigma_{ij} = \sum_{k,l} \mathcal{C}_{ijkl}\, \epsilon_{kl}, \qquad \boldsymbol{\sigma} = \mathcal{C}\,[\epsilon]\,,}$$

The elastic moduli $\mathcal{C}_{ijkl}$ possess the symmetries

$$\begin{aligned} \mathcal{C}_{ijkl} &= \mathcal{C}_{klij}, \\ \mathcal{C}_{ijkl} &= \mathcal{C}_{jikl}, \\ \mathcal{C}_{ijkl} &= \mathcal{C}_{ijlk}. \end{aligned}$$

For the most general linear elastic material there are 21 independent elastic moduli $\mathcal{C}_{ijkl}$.

The **stiffness tensor** $\mathcal{C}$ is said to be **positive definite** if

$$\sum_{i,j,k,l} \mathcal{C}_{ijkl}\epsilon_{ij}\epsilon_{kl} > 0, \quad \text{for all} \quad \epsilon \neq \mathbf{o}$$

We assume that the $\mathcal{C}$ is positive definite. Physically, we assume that the strain energy density W is positive valued, whenever the strain is non-zero. In this case the stress-strain relation

$$\boxed{\sigma_{ij} = \sum_{k,l} \mathcal{C}_{ijkl}\epsilon_{kl}}$$

is invertible, such that

$$\boxed{\epsilon_{ij} = \sum_{k,l} \mathcal{S}_{ijkl}\sigma_{kl}.}$$

The fourth order tensor $\mathcal{S}$ is called the **compliance tensor**.

MATERIAL SYMMETRY

Most solids exhibit symmetry properties with respect to certain rotations of the body, or reflection about one or more planes. The effects of these symmetries is to reduce the number of elastic constants from the number 21 for the most general anisotropic material.

Recall that if $\sigma_{ij} = \mathbf{e}_i \cdot \sigma \mathbf{e}_j$ and $\sigma'_{ij} = \mathbf{e}'_i \cdot \sigma \mathbf{e}'_j$ are the components of the stress tensor $\boldsymbol{\sigma}$ with respect to the two bases $\{\mathbf{e}_i\}$ and $\left\{\mathbf{e}'_i\right\}$, then σ'_{ij} and σ_{ij} are related by

$$\sigma'_{ij} = \sum_{k,l} Q_{ik} Q_{jl} \sigma_{kl}.$$

This is the tensor transformation law for 2nd-order tensors.

If $\mathcal{S}_{ijkl}$ and $\mathcal{S}'_{ijkl}$ are the components of the compliance tensor $\boldsymbol{\mathcal{S}}$ with respect to the two bases, then they are related by the tensor transformation law for 4th-order tensors

$$\mathcal{S}'_{ijkl} = \sum_{p,q,r,s} Q_{ip} Q_{jq} Q_{kr} Q_{ls} \mathcal{S}_{pqrs}$$

Note that these ransformation laws hold for **all** orthogonal matrices $[Q]$.

If for some $[Q]$ the values of the compliance coefficients in the primed system are the same as those in the unprimed system,

$$\mathcal{S}'_{ijkl} = \mathcal{S}_{ijkl},$$

or equivalently,

$$\mathcal{S}_{ijkl} = \sum_{p,q,r,s} Q_{ip} Q_{jq} Q_{kr} Q_{ls} \mathcal{S}_{pqrs},$$

that is if the material properties in the two bases have the same values, then $[Q]$ is called a **symmetry transformation**.

Isotropy

A material is said to be elastically **isotropic** if its elastic modulii are invariant with respect to all orthogonal transformations. That is,

$$\mathcal{S}_{ijkl} = \sum_{p,q,r,s} Q_{ip} Q_{jq} Q_{kr} Q_{ls} \mathcal{S}_{pqrs}$$

holds for **all** orthogonal matrices $[Q]$.

Working out the details, it may be shown that an **isotropic** linear elastic material has only two **2 independent elastic constants**, and that we may write the strain-stress relation $\epsilon_{ij} = \sum_{k,l} \mathcal{S}_{ijkl}\sigma_{kl}$ in matrix form as

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{1111} & \mathcal{S}_{1122} & \mathcal{S}_{1122} & 0 & 0 & 0 \\ \mathcal{S}_{1122} & \mathcal{S}_{1111} & \mathcal{S}_{1122} & 0 & 0 & 0 \\ \mathcal{S}_{1122} & \mathcal{S}_{1122} & \mathcal{S}_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix},$$

with

$$\xi = 2\left(\mathcal{S}_{1111} - \mathcal{S}_{1122}\right).$$

Physical Interpretation of Elastic Moduli For an Isotropic Material

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{1111} & \mathcal{S}_{1122} & \mathcal{S}_{1122} & 0 & 0 & 0 \\ \mathcal{S}_{1122} & \mathcal{S}_{1111} & \mathcal{S}_{1122} & 0 & 0 & 0 \\ \mathcal{S}_{1122} & \mathcal{S}_{1122} & \mathcal{S}_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}, \quad \xi = 2\left(\mathcal{S}_{1111} - \mathcal{S}_{1122}\right).$$

Consider a uniaxial stress situation $\sigma_{11} \neq 0$, all other $\sigma_{ij} = 0$. Then

$$\epsilon_{11} = \mathcal{S}_{1111}\sigma_{11}, \quad \epsilon_{22} = \epsilon_{33} = \mathcal{S}_{1122}\sigma_{11}.$$

Defining the **Young's Modulus**, E, and the **Poisson's ratio**, ν, by

$$\boxed{E \equiv \frac{\sigma_{11}}{\epsilon_{11}}, \quad \text{and} \quad \nu \equiv -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}},}$$

we have

$$E = \frac{1}{\mathcal{S}_{1111}}, \quad \text{and} \quad \nu = -\frac{\mathcal{S}_{1122}}{\mathcal{S}_{1111}}.$$

Next consider a pure shear stress $\sigma_{12} \neq 0$, all other $\sigma_{ij} = 0$. Then

$$2\epsilon_{12} = 2\left(\mathcal{S}_{1111} - \mathcal{S}_{1122}\right)\sigma_{12} = \frac{2\left(1+\nu\right)}{E}\sigma_{12}$$

Then, defining the **Shear Modulus**, G, for an isotropic material by

$$\boxed{G \equiv \frac{\sigma_{12}}{2\epsilon_{12}},}$$

we have

$$\boxed{G \equiv \frac{E}{2\left(1+\nu\right)}}$$

Thus

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}.$$

This strain stress relation may be written in indicial notation as

$$\epsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\left(\sum_k \sigma_{kk}\right)\delta_{ij},$$

where the two-independent elastic constants are the **Young's modulus**, E, and the **Poisson's ratio** ν.

The shear modulus G for an isotropic elastic material is given by

$$G \equiv \frac{E}{2\,(1+\nu)}$$

Next consider a state of **hydrostatic pressure**

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \text{all other} \quad \sigma_{ij} = 0.$$

In this case

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \frac{1-2\nu}{E}(-p), \quad \text{all other} \quad \epsilon_{ij} = 0,$$

and

$$\sum_k \epsilon_{kk} = \frac{3(1-2\nu)}{E}(-p), \quad \frac{1}{3}\sum_k \sigma_{kk} = -(p)$$

The **bulk modulus** for an isotropic material is defined by

$$K \equiv \frac{\frac{1}{3}\sum_k \sigma_{kk}}{\sum_k \epsilon_{kk}} = \frac{\text{mean normal pressure}}{\text{volume change}}.$$

Thus

$$\boxed{K \equiv \frac{E}{3(1-2\nu)}}$$

Considerations of the positive definiteness of the strain energy density W lead to the restrictions

$$\boxed{E > 0, \quad 0 < \nu < \frac{1}{2}, \quad G > 0, \quad K > 0.}$$

Constitutive Relation For Linear Elasticity in terms of E and ν:

$$\boxed{\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\ \left(\sum_k \sigma_{kk}\right)\ \delta_{ij}\right]}$$

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\ \left(\sum_k \epsilon_{kk}\right)\ \delta_{ij}\right]}$$

Constitutive Relation For Linear Elasticity in terms of G and K:

$$\sigma'_{ij} = 2G\,\epsilon'_{ij},$$

$$\frac{1}{3}\left(\sum_k \sigma_{kk}\right) = K\,\left(\sum_k \epsilon_{kk}\right)$$

$$\boxed{\epsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \left(\frac{3K-2G}{9K}\right)\,\left(\sum_k \sigma_{kk}\right)\,\delta_{ij}\right].}$$

$$\boxed{\sigma_{ij} = 2G\epsilon_{ij} + (K - \frac{2}{3}G)\,\left(\sum_k \epsilon_{kk}\right)\,\delta_{ij}.}$$

Relations Between Elastic Constants

	G	K	E	ν
$G,\ E$		$\frac{GE}{3(3G-E)}$		$\frac{E-2G}{2G}$
$G,\ \nu$		$\frac{2G(1+\nu)}{3(1-2\nu)}$	$2G(1+\nu)$	
$G,\ K$			$\frac{9KG}{3K+G}$	$\frac{1}{2}\left[\frac{3K-2G}{3K+G}\right]$
$E,\ \nu$	$\frac{E}{2(1+\nu)}$	$\frac{E}{3(1-2\nu)}$		
$E,\ K$	$\frac{3EK}{9K-E}$			$\frac{1}{2}\left[\frac{3K-E}{3K}\right]$
$\nu,\ K$	$\frac{3K(1-2\nu)}{2(1+\nu)}$		$3K(1-2\nu)$	

For metallic materials it is commonly found that

$$\nu \approx \frac{1}{3}, \quad \Longrightarrow \quad G \approx \frac{3}{8}E, \qquad K \approx E.$$

Thermal strains

In the absence of stress, the strain caused by a small change in temperature from T_0 in the reference configuration to T in the current configuration is called the **thermal strain**. These strains are expressed by the linear relation

$$\epsilon_{ij}^{\text{thermal}} = A_{ij}\,(T - T_0)$$

where A_{ij} is called the **thermal expansion tensor**.

For **isotropic materials**,

$$A_{ij} = \alpha\delta_{ij},$$

where α is called the **coefficient of thermal expansion**.

Thermo-Elasticity For Isotropic Materials

$$\epsilon_{ij}^{\mathrm{thermal}} = \alpha\ (T - T_0)\ \delta_{ij}$$

$$\epsilon_{ij}^{\mathrm{mechanical}} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\ \left(\sum_k \sigma_{kk}\right)\ \delta_{ij}\right]$$

For the case of both an application of stress and a change in temperature, the thermo-elastic strains in a linear theory are written as

$$\epsilon_{ij} = \epsilon_{ij}^{\mathrm{mechanical}} + \epsilon_{ij}^{\mathrm{thermal}}.$$

Hence,

$$\boxed{\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\left(\sum_k \sigma_{kk}\right)\delta_{ij}\right] + \alpha\,(T - T_0)\,\delta_{ij},}$$

which can be easily inverted to give

$$\boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\left(\sum_k \epsilon_{kk}\right)\ \delta_{ij} - \frac{(1+\nu)}{(1-2\nu)}\alpha\,(T - T_0)\,\delta_{ij}\right].}$$

Failure/Yield Condition

In addition to the small displacement gradient and small temperature change restrictions in the theory of linear elasticity, we need to also explicitly introduce a **failure criterion** which bounds the levels of stresses beyond which the constitutive equation for isotropic linear elasticity is no longer valid.

For **isotropic materials**, a simple statement of a failure condition is

$$f(\boldsymbol{\sigma}) \leq \sigma_f,$$

where $f(\boldsymbol{\sigma})$ is a scalar-valued function of the applied stress $\boldsymbol{\sigma}$, and the scalar number σ_f is a **material property** called the **strength** of the material.

Isotropy requires that the dependence on $\boldsymbol{\sigma}$ in the function $f(\boldsymbol{\sigma})$ can only appear in terms of its **invariants**,

$$f(\text{invariants of } \boldsymbol{\sigma}) \leq \sigma_f.$$

or equivalently, in terms of the **principal values** of the stress,

$$f(\sigma_1, \sigma_2, \sigma_3) \leq \sigma_f.$$

For ductile metallic polycrystalline materials, "failure" of the elastic response occurs when dislocations move large distances through the crystals of a material to produce significant permanent deformation. Thus, for metallic materials the "failure criterion" is actually a "yield criterion."

It has been found experimentally that for the yielding mechanism in metallic materials the yield function $f(\text{invariants of } \boldsymbol{\sigma})$ may be approximated as

$$f(\text{invariants of } \boldsymbol{\sigma}) = f(\sigma_1, \sigma_2, \sigma_3) \approx \bar{\sigma},$$

where

$$\begin{aligned}
\bar{\sigma} &= \sqrt{(3/2) \sum_{i,j} \sigma'_{ij} \sigma'_{ij}}, \\
&= \left| \left[\frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} \right.\right. \\
&\qquad \left.\left. + \, 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right]^{1/2} \right|, \\
&= \left| \left[\frac{1}{2} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} \right]^{1/2} \right|,
\end{aligned}$$

is called the **equivalent tensile stress**.

Thus, the **yield condition** for ductile metallic materials may be written as

$$\boxed{\bar{\sigma} \leq \sigma_y,}$$

where σ_y is the **tensile yield strength** of the material.

This yield condition was first proposed by Richard von Mises in 1913, and is known as the **Mises yield condition**, and $\bar{\sigma}$ is called the **Mises stress**.

This yield condition stands for the physical notion that, as long as the equivalent tensile stress $\bar{\sigma}$ applied on a material is less than the **material property** σ_y, dislocations would not have moved large distances through the crystals of a polycrystalline material to have produced significant permanent deformation.

The **strength** σ_y is typically identified with the 0.2% offset **yield strength** in a tension (or compression) test and is defined as the stress level from which unloading to zero stress would result in a permanent axial strain of 0.2%.

We will discuss failure/yield conditions for other material classes in a later part of our study on Mechanics of Materials.

Tresca Yield Condition

In terms of the principal stresses, the maximum shear stress in a material at a given point is given by $\frac{1}{2}(\sigma_1 - \sigma_3)$, and as early as 1864, Henri Edouard Tresca had proposed the yield condition

$$\frac{1}{2}\left|\sigma_1 - \sigma_3\right| \leq \tau_y$$

for metallic materials, where τ_y is the **yield strength in shear**.

In general three–dimensional formulations, there are some mathematical difficulties associated with plasticity theories based on the Tresca yield criterion. It is for this reason that the mathematically more tractable theories of plasticity based on the Mises yield condition are in more wide use these days.

SUMMARY

Limiting ourselves to **isothermal** situations, we record that the three-dimensional theory of **isotropic** linear elasticity is based on:

1. **The Strain-Displacement Relations**

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \qquad \epsilon_{ij} = \epsilon_{ji}, \quad \left|\frac{\partial u_i}{\partial x_j}\right| << 1.$$

2. **The Stress-Strain Relations**

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\left(\sum_k \epsilon_{kk}\right)\delta_{ij}\right],$$

subject to the **yield condition**

$$\bar{\sigma} \leq \sigma_y,$$

with

$$\begin{aligned}
\bar{\sigma} &= \sqrt{(3/2)\sum_{i,j}\sigma'_{ij}\sigma'_{ij}}, \\
&= \left| \left[\frac{1}{2}\left\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right\} \right.\right. \\
&\qquad \left.\left. + 3\left\{\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right\}\right]^{1/2} \right|, \\
&= \left| \left[\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}\right]^{1/2} \right|,
\end{aligned}$$

3. **The Equations of Motion**

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2}{\partial t^2} u_i \qquad (i = 1, 2, 3).$$

4. **Plus Appropriate Boundary Conditions For Surface Tractions and Displacements.**

- There is a vast amount of engineering, scientific and mathematical literature associated with particular solutions to this set of equations, obtained by using specialized analytical techniques. Since about the end of the 1970's, the digital computer revolution and the associated development of the computational technique called the **finite element method** have made a major change in how these equations are solved in engineering practice.

- The availability of software incorporating the finite element method and other procedures for solid-modeling and post-processing of results has placed the advanced concepts of elasticity into the hands of a broad community of engineers.

- At the same time, it has created a necessity for them to have a much deeper education and a strong foundation in the underlying physical and mathematical basis of the theory, so that the new computational techniques are used properly to reliably interpret and assess the quality of the approximate solutions they provide.

METALS		E (GPa)	ν	G (GPa)	α $(10^{-6}/K)$
Tungsten	W	397	0.284	153	4.3 – 4.7
Molybdenum	Mo	327	0.30	116	4.9
Chromium	Cr	243	0.209	117	6.2
Iron	Fe	210	0.279	82	10.6–12.8
Nickel	Ni	193	0.3	75	12.5
Copper	Cu	124	0.345	45	16.5
Titanium	Ti	106	0.345	39	8.6
Zinc	Zn	92	0.29	37	30.0
Silver	Ag	81	0.37	29	20.0
Gold	Au	78	0.425	28	13.0
Aluminum	Al	71	0.34	27	23.2
Tin	Sn	53	0.375	19	23
Magnesium	Mg	44	0.28	17	26.1
Lead	Pb	16	0.44	5.4	29.3

CERAMICS	E (GPa)	ν	G (GPa)	α $(10^{-6}/K)$
Diamond	1128	0.18	451	1.2
Metal-bonded Tungsten Carbide				
94 WC, 6 Co	580	0.26	230	
Self-bonded Silicon Carbide				
90 SiC, 10 Si	410	0.24	165	4.3
Sintered Alumina				
100 Al_2O_3	350	0.23	142	8.5
Hot-pressed Silicon Nitride				
96 Si_3N_4, 4MgO	310	0.25	124	3.2
Low-expansion Glass Ceramic				
2 (Ti, Zr) O_2, 4 Li_2O				
20 Al_2O_3, 70 SiO_2	87	0.25	35	
Soda-Lime Glass				
13 Na_2O, 12(Ca, Mg)O,				
72 SiO_2	73	0.21	30	8.5
Vitreous Silica 100 SiO_2	71	0.17	30	
Low-expansion Borosilicate Glass				
12 B_2O_3, 4 Na_2O, 2 Al_2O_3, 80 SiO_2	66	0.2	27.5	4.0
Machineable Glass Ceramic				
65 Mica, 35 Glass	64	0.26	25	
High-density Molded Graphite	9	0.11	4	

POLYMERIC MATERIALS		E (GPa)	ν	G (GPa)	α $(10^{-6}/K)$
Polymethylmethacrylate					
PMMA	-125°C	6.3	0.26	2.5	
	25°C	3.7	0.33	1.39	54-72
Polystyrene					
PS	25°C	3.4	0.33	1.28	70-100
Polyethylene					
(low density)	25°C	2.4	0.38	0.87	160-190
Polycarbonate					
PC	25°C	2.3	0.2	0.96	
Polyethylene terephthalate					
PET	25°C	2	0.35	0.74	
Polyamide (nylon)					
PA	25°C	2.8	0.4	1.0	80-95
Vulcanized Natural Rubber					
VNR	25°C	0.0016	0.499	0.0005	600
Polyurethane Foam Rubber					
EUFR	25°C	0.0005	0.25	0.0002	600