Euler-Bernoulli Beams: Bending, Buckling, and Vibration

David M. Parks

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Department of Mechanical Engineering
MIT
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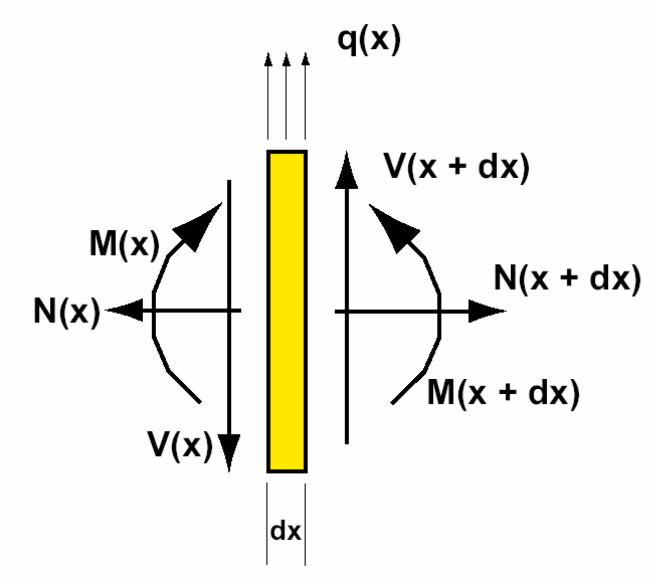
Linear Elastic Beam Theory

- Basics of beams
  - Geometry of deformation
  - Equilibrium of “slices”
  - Constitutive equations
- Applications:
  - Cantilever beam deflection
  - Buckling of beams under axial compression
  - Vibration of beams

Beam Theory:
Slice Equilibrium Relations

- **q(x): distributed load/length**
- **N(x): axial force**
- **V(x): shear force**
- **M(x): bending moment**



Axial force balance:

$$0 = N(x+dx) - N(x) \Rightarrow N(x) = \text{constant}$$

Transverse force balance:

$$\begin{aligned} 0 &= q(x)dx + V(x+dx) - V(x) \\ &= q(x)dx + \Big(V(x) + V'(x)dx + o(dx)\Big) - V(x) \\ &= dx\left[V'(x) + q(x)\right] \Rightarrow \\ 0 &= V'(x) + q(x) \quad \text{CDL}(3.11) \end{aligned}$$

Moment balance about 'x+dx':

$$\begin{aligned} 0 &= V(x)dx + M(x+dx) - M(x) - (q(x)dx)\ dx \\ &= V(x)dx + \Big(M(x) + M'(x)dx\Big) - M(x) - (q(x)dx)\ dx/2 \\ &= dx\left[M'(x) + V(x) - q(x)dx/2\right] \Rightarrow \\ 0 &= M'(x) + V(x) \quad \text{CDL}(3.12) \end{aligned}$$

Euler-Bernoulli Beam Theory: Displacement, strain, and stress distributions

Beam theory assumptions on spatial variation of displacement components:

$$\begin{aligned} u(x,y,z) &= u_0(x) - y v'(x) \\ v(x,y,z) &= v(x) \\ w(x,y,z) &= 0 \end{aligned}$$

1-D stress/strain relation:

$$\sigma_{xx} = E \epsilon_{xx}$$

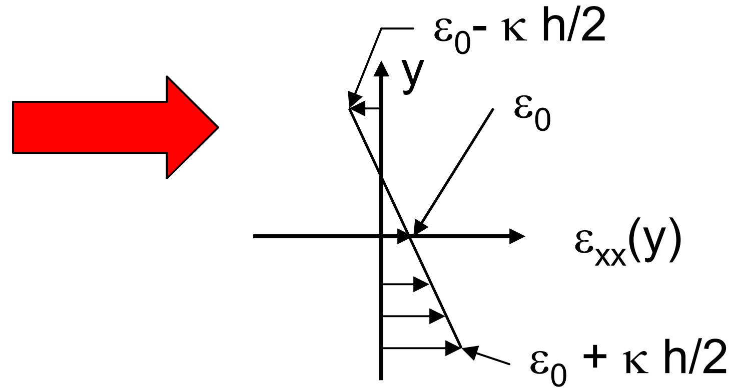
Stress distribution in terms of Displacement field:

$$\sigma_{xx}(x,y,z) = E\,(\epsilon_0(x) - y\kappa(x))$$

Axial strain distribution in beam:

$$\begin{aligned} \epsilon_{xx}(x,y,z) &\equiv \frac{\partial u(x,y,z)}{\partial x} \\ &= u_0'(x) - y v''(x) \\ &\equiv \epsilon_0(x) - y\kappa(x) \end{aligned}$$

<u>Axial **strain** varies linearly Through-thickness at section 'x'</u>



Slice Equilibrium: Section Axial Force N(x) and Bending Moment M(x) in terms of Displacement fields

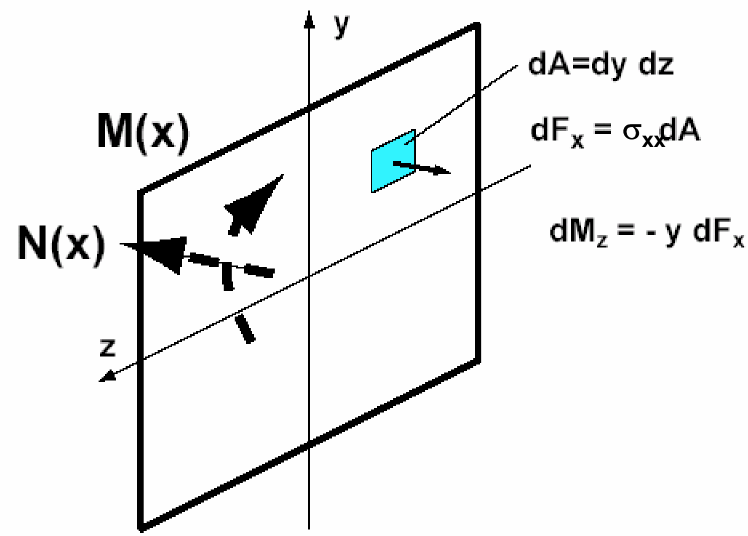
N(x): x-component of force equilibrium on slice at location ‘x’:

$$
\begin{aligned}
N(x) &\equiv \int \sigma_{xx}(x,y,z)\, dA(y,z) \\
&= \int E\left\{\epsilon_0(x) - y\kappa(x)\right\} dA \\
&= EA\epsilon_0(x) - E\kappa(x)\int y\, dA.
\end{aligned}
$$

M(x): z-component of moment equilibrium on slice at location ‘x’:

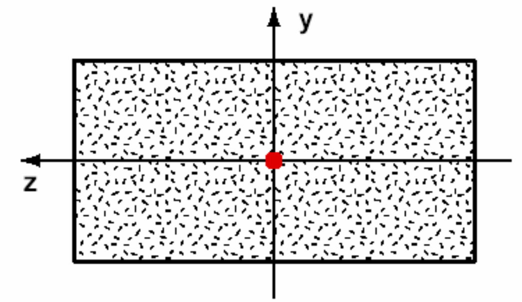
$$
\begin{aligned}
M(x) &\equiv \int -y\, \sigma_{xx}(x,y,z)\, dA(y,z) \\
&= \int E\left\{-y\epsilon_0(x) + y^2\kappa(x)\right\} dA \\
&= -E\epsilon_0(x)\int y dA + E\kappa(x) I
\end{aligned}
$$

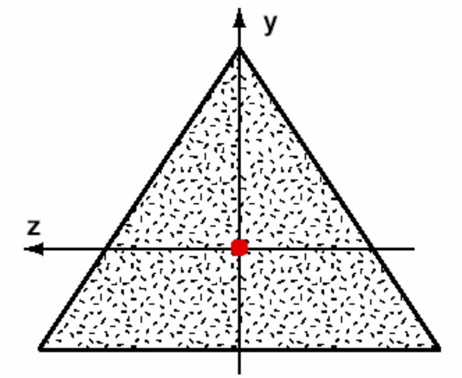
where $I \equiv \int y^2\, dA$ is area moment of inertia of cross section



Centroidal Coordinates

$$\bar{y} \;=\; \frac{1}{A}\int y\,dA$$





choice: $\bar{y} \equiv 0 \Rightarrow \int y\,dA = 0$

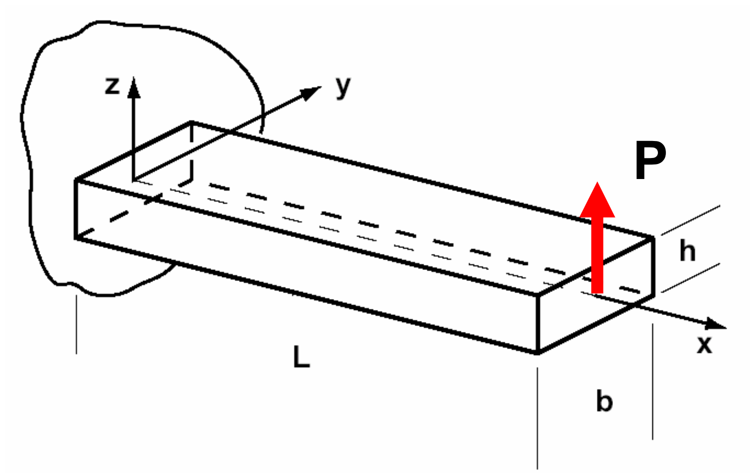
Simplifications:

$$\begin{aligned} N(x) &= EA\epsilon_0(x) = EAu_0'(x) \\ M(x) &= EI\kappa(x) = EIv''(x) \end{aligned}$$

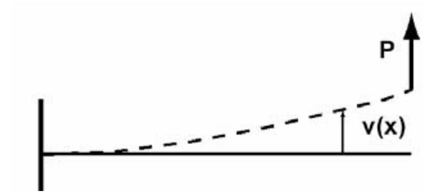
Note: I is centroidal area moment of inertia:

$$I \equiv \int y^2\,dA$$

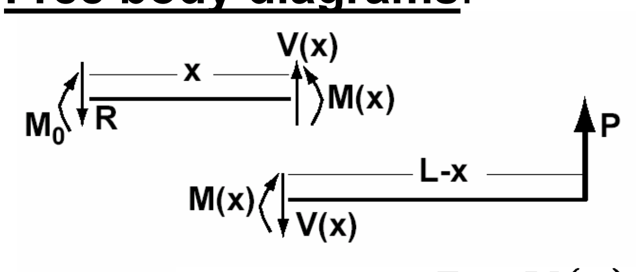
Tip-Loaded Cantilever Beam: Equilibrium



•statically determinant: support reactions R, M_0 from equilibrium alone
•reactions "present" because of x=0 geometrical boundary conditions v(0)=0; v'(0)=ϕ(0)=0

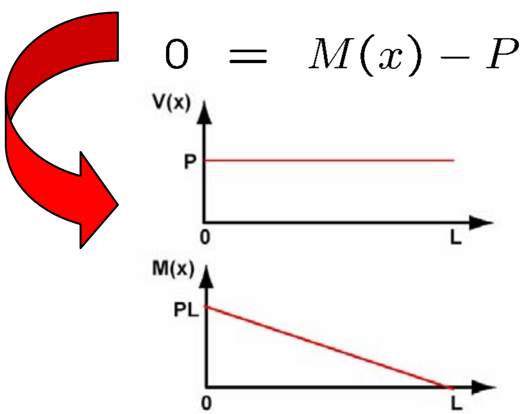


Free body diagrams:



$$0 = P - V(x)$$

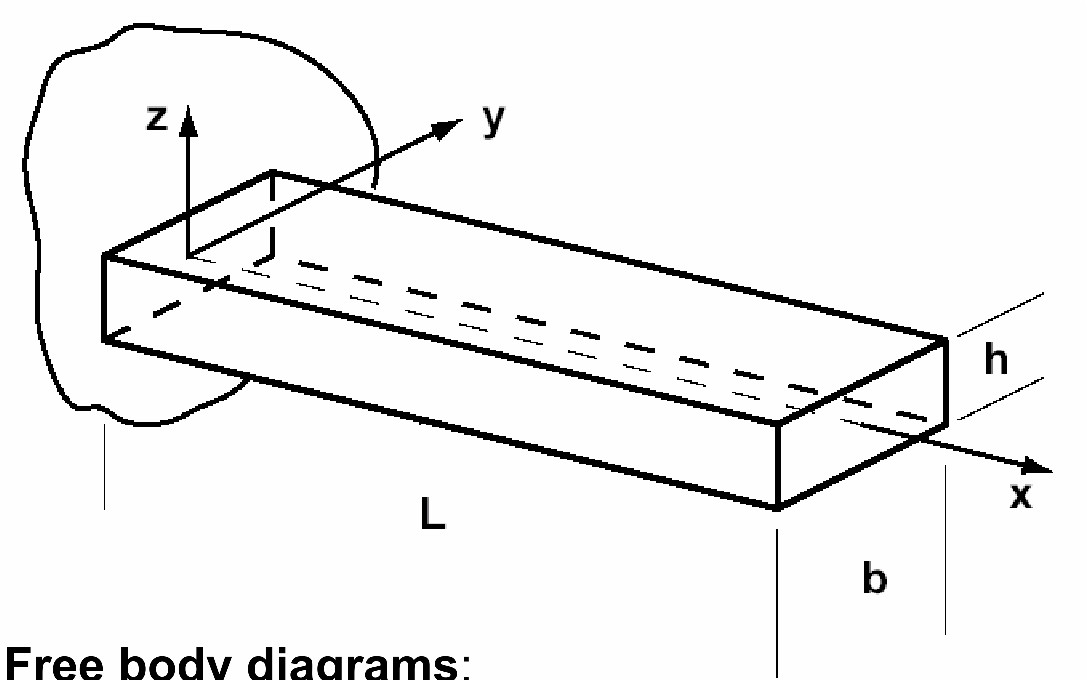
$$0 = M(x) - P(L-x)$$



•general equilibrium equations (CDL 3.11-12) satisfied

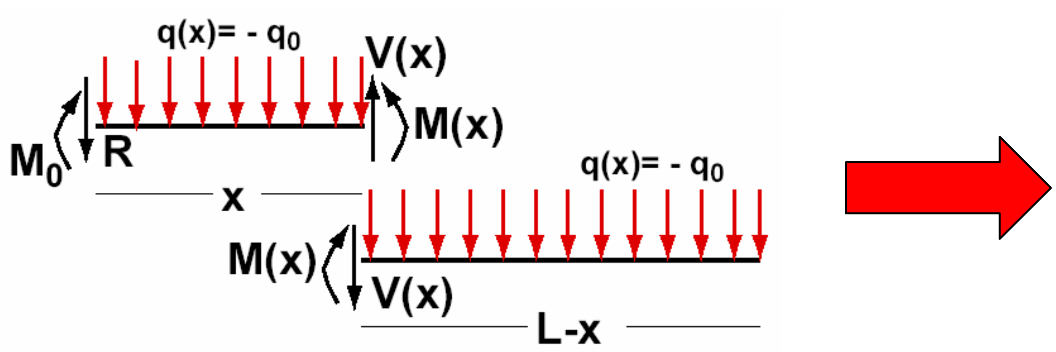
How to determine lateral displacement v(x); especially at tip (x=L)?

Exercise: Cantilever Beam Under Self-Weight



- Weight per unit lenth: q_0
- $q_0 = \rho A g = \rho b h g$

Free body diagrams:



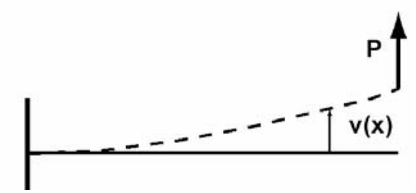
Find:

- Reactions: R and M_0
- Shear force: V(x)
- Bending moment: M(x)

Tip-Loaded Cantilever: Lateral Deflections

curvature / moment relations:

$$\begin{aligned} v''(x) &= \frac{1}{EI} M(x) \\ &= \frac{1}{EI} \left(P(L-x)\right) \Rightarrow \\ v'(x) &= \frac{P}{EI} \left(Lx - x^2/2 + C_1\right) \Rightarrow \\ v(x) &= \frac{P}{EI} \left(Lx^2/2 - x^3/6 + C_1 x + C_2\right) \end{aligned}$$



geometric boundary conditions

$$\begin{aligned} \phi(0) = v'(0) &= 0 \Rightarrow C_1 = 0 \\ v(0) &= 0 \Rightarrow C_2 = 0 \\ v(x) &= \frac{Px^2}{6EI} (3L - x) \end{aligned}$$

tip deflection and rotation:

$$\begin{aligned} \Delta \equiv v(L) &= \frac{PL^3}{3EI} \\ \Phi \equiv v'(L) &= \frac{PL^2}{2EI} \end{aligned}$$

stiffness and modulus:

$$\begin{aligned} k \equiv \frac{P}{\Delta} &= \frac{3EI}{L^3} \\ E &= \frac{kL^3}{3I} = \frac{PL^3}{3I\Delta} \end{aligned}$$

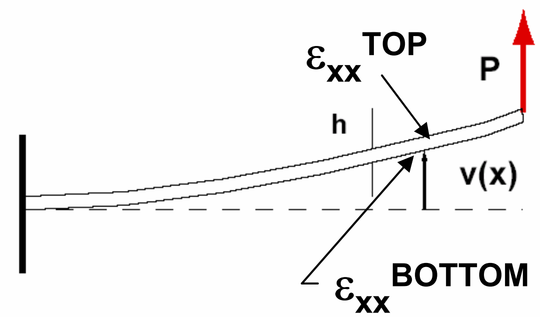
Tip-Loaded Cantilever: Axial Strain Distribution

strain field (no axial force):

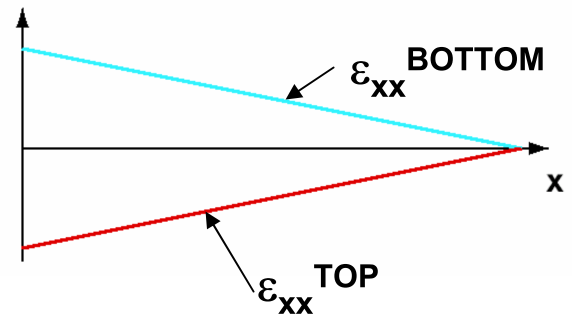
$$\begin{aligned}\epsilon_{xx}(x,y) &= -yv''(x)\\ &= -\frac{yM(x)}{EI}\end{aligned}$$

top/bottom axial strain distribution:

$$\begin{aligned}\epsilon_{xx}^{TOP}(x) &= -\frac{6P(L-x)}{bh^2E} \quad (y=h/2)\\ \epsilon_{xx}^{BOTTOM}(x) &= \frac{6P(L-x)}{bh^2E} \quad (y=-h/2)\end{aligned}$$



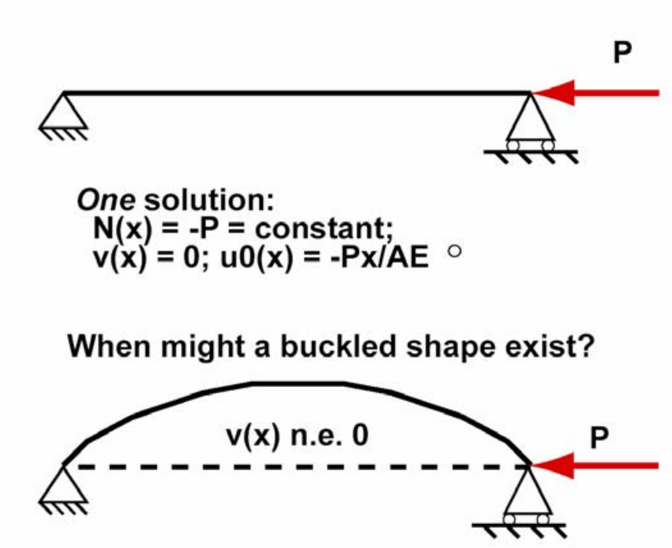
$$I_{\text{rectangle}} = \frac{bh^3}{12}$$



strain-gauged estimate of E:

$$E = \frac{6P(L-x)}{bh^2\epsilon_{xx}^{BOTTOM}(x)} = \frac{6P(L-x)}{bh^2|\epsilon_{xx}^{TOP}(x)|}$$

Euler Column Buckling: Non-uniqueness of deformed configuration



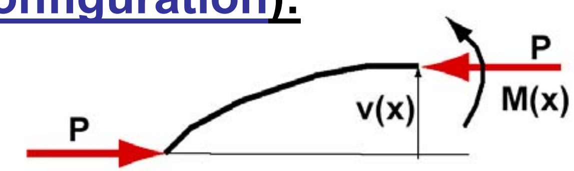
moment/curvature:

$$M(x) = EI\kappa(x) = EIv''(x)$$

ode for buckled shape:

$$\begin{aligned} 0 &= M(x) + Pv(x) \\ &= EIv''(x) + Pv(x) \\ 0 &= v''(x) + \frac{P}{EI}v(x) \\ &\equiv v''(x) + k^2 v(x) \end{aligned}$$

free body diagram (note: evaluated in deformed configuration):



$$\sum M_z = 0 \Rightarrow M(x) + Pv(x) = 0$$

Note: linear 2nd order ode; Constant coefficients (but parametric: $k^2 = P/EI$

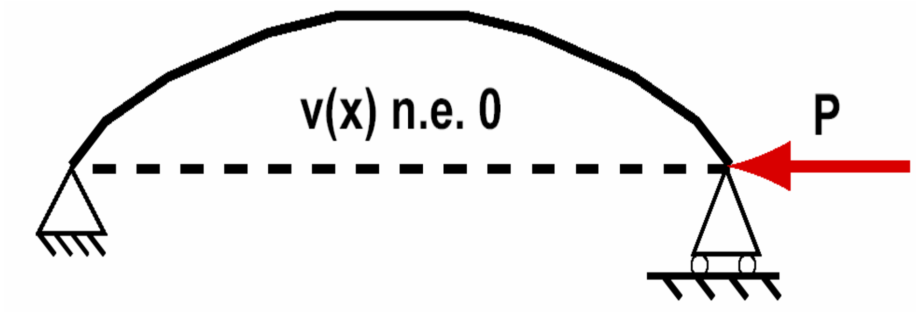
Euler Column Buckling, Cont.

<u>ode for buckled shape:</u>

$$0 = v''(x) + k^2 v(x)$$

<u>general solution to ode:</u>

$$v(x) = C_1 \sin kx + C_2 \cos kx$$



<u>boundary conditions:</u>

$$v(0) = 0 \Rightarrow C_2 = 0$$

$$v(L) = 0 \Rightarrow C_1 \sin kL = 0 \Rightarrow$$

$$C_1 = 0 \, (\text{trivial}) \quad \text{or} \sin kL = 0$$

<u>parametric consequences:</u>
<u>non-trivial buckled shape only when</u>

$$\sin kL = 0 \Rightarrow kL = n\pi$$

$$k^2 = (n\pi/L)^2$$

$$P = EIk^2 = \frac{n^2 EI \pi^2}{L^2}$$

$$P_{\text{crit}}(n = 1) = \frac{\pi^2 EI}{L^2}$$

<u>buckling-based estimate of E:</u>

$$E_{\text{pinned/pinned}} = \frac{P_{\text{crit}} L^2}{\pi^2 I}$$

Euler Column Buckling: General Observations

- buckling load, P_{crit}, is proportional to EI/L^2

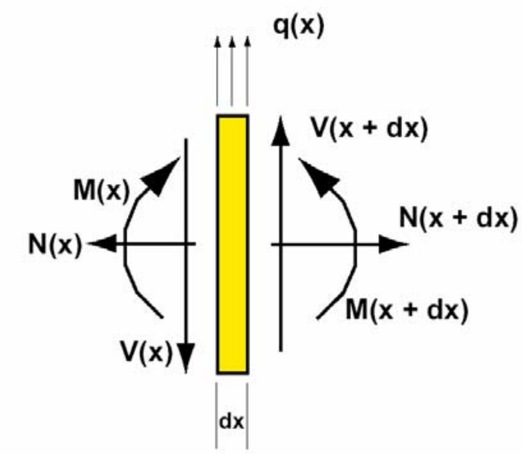
- proportionality constant depends <u>strongly</u> on boundary conditions at both ends:

- the more kinematically restrained the ends are, the larger the constant and the higher the critical buckling load (see Lab 1 handout)

- safe design of long slender columns requires adequate margins with respect to buckling

- buckling load may occur a a compressive stress value (σ=P/A) that is less than yield stress, σ_y

Euler-Bernoulli Beam Vibration



assume time-dependent lateral motion:

$$v(x,t) = \bar{v}(x)\,\sin\omega t$$

lateral velocity of slice at 'x':

$$\frac{\partial v(x,t)}{\partial t} \equiv \dot{v}(x,t) = \omega\,\bar{v}(x)\,\cos\omega t$$

lateral acceleration of slice at 'x':

$$\frac{\partial^2 v(x,t)}{\partial t^2} \equiv \ddot{v}(x,t) = -\omega^2\,\bar{v}(x)\,\sin\omega t$$

mass of dx-thickness slice:

$$dm = dx\,\rho A$$

linear momentum balance (Newton):

$$\begin{aligned}\sum F_y &= dm\,\ddot{v}(x,t) \Rightarrow \\ 0 &= dx\left(M''(x,t) - \rho\omega^2 A\bar{v}(x)\sin\omega t\right)\end{aligned}$$

net lateral force (q(x,t)=0):

$$\sum F_y = dx\,\frac{\partial V(x,t)}{\partial x} \equiv dx\,V'(x,t)$$

moment balance:

$$\begin{aligned}0 &= \frac{\partial M(x,t)}{\partial x} + V(x,t) \\ &\equiv M'(x,t) + V(x,t) \Rightarrow \\ 0 &= M''(x,t) + V'(x,t)\end{aligned}$$

Euler-Bernoulli Beam Vibration, Cont.(1)

linear momentum balance:

$$0 = M''(x,t) - \rho\omega^2 A\bar{v}(x)\sin\omega t$$

moment/curvature:

$$M(x,t) = EI\,\kappa(x,t) = EI\bar{v}''(x)\sin\omega t$$

ode for mode shape, v(x), and vibration frequency, ω:

$$\begin{aligned} 0 &= M''(x,t) - \rho\omega^2 A\bar{v}(x)\sin\omega t \\ &= \sin\omega t\left(EI\bar{v}''''(x) - \rho\omega^2 A\bar{v}(x)\right) \\ &= \sin\omega t\left(\bar{v}''''(x) - \frac{\rho\omega^2 A}{EI}\bar{v}(x)\right) \\ &\equiv \sin\omega t\left(\bar{v}''''(x) - \beta^4\bar{v}(x)\right) \Rightarrow \\ 0 &= \bar{v}''''(x) - \beta^4\bar{v}(x) \end{aligned}$$

general solution to ode:

$$\bar{v}(x) = A_1\sin\beta x + A_2\cos\beta x + A_3\sinh\beta x + A_4\cosh\beta x$$

Euler-Bernoulli Beam Vibration, Cont(2)

general solution to ode:

$$\bar{v}(x) = A_1 \sin \beta x + A_2 \cos \beta x + A_3 \sinh \beta x + A_4 \cosh \beta x$$

pinned/pinned boundary conditions:

$$\bar{v}(0) = 0 \quad \Rightarrow \quad A_2 + A_4 = 0$$
$$\bar{v}''(0) = 0 \quad \Rightarrow \quad \beta^2 (-A_2 + A_4) = 0$$
$$\bar{v}(L) = 0 \quad \Rightarrow \quad A_1 \sin \beta L + A_3 \sinh \beta L = 0$$
$$\bar{v}''(L) = 0 \quad \Rightarrow \quad \beta^2 (-A_1 \sin \beta L + A_3 \sinh \beta L) = 0$$

pinned/pinned restricted solution:

$$\beta \neq 0; \qquad A_2 = A_3 = A_4 = 0;$$

$$A_1 \sin \beta L = 0 \Rightarrow$$

$$A_1 = 0 \, (\text{trivial}), \qquad \text{OR}$$

$$\sin \beta L = 0 \Rightarrow \beta L = n\pi$$

Solution (n=1, first mode):

A_1: 'arbitrary' (but small) vibration amplitude

$$\beta_1 = \pi / L \Rightarrow$$

$$v_{(n=1)}(x,t) = A_1 \, \sin(\pi x / L) \, \sin \omega_1 t$$

$$\beta_1^4 = (\pi / L)^4 = \omega_1^2 \rho A / EI \Rightarrow$$

$$\omega_1 = \sqrt{\frac{EI\pi^4}{\rho A L^4}}$$

τ_1: period of first mode:

$$\tau_1 = \frac{2\pi}{\omega_1}$$