Yielding Under Multi-axial Stress and Elastic-Plastic Stress-Strain Relations

2.002 Mechanics and Materials II
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Uniaxial tension/compression:
initial linear elastic response,
as axial stress, σ, is increased
up to the uniaxial "yield condition":

$$|\sigma| \leq \sigma_y$$

Suppose that, at some location
in a body made of the same material,
the state of stress is multi-axial, with
cartesian components σ_{ij};

QUESTION: Will plastic deformation occur under this state of stress?

Approach: we need to define a non-negative scalar, stress-valued function of [all] the stress components, such that it can consistently generalize the uniaxial yield criterion, $|\sigma| < \sigma_y$

Observation # 1: pressure <u>in</u>sensitivity of uniaxial yielding

Suppose that a uniaxial test is performed under fixed superposed hydrostatic pressure, *p,* so the cartesian stress components are

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma - p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Plastic deformation is observed to commence when $|\sigma| = \sigma_y$, essentially independent of the value of *p*

This suggests that yielding is ~ independent of the mean normal stress given by $\Sigma = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$

Recall the stress deviator tensor, whose components are given by

$$\left[\sigma_{ij}^{(\text{dev})}\right] \equiv \left[\sigma_{ij}\right] - \frac{1}{3}\left(\sum_{k=1}^{3} \sigma_{kk}\right)\left[\delta_{ij}\right]$$

Clearly, *the stress deviator tensor is independent of the mean normal stress*

The **Mises equivalent tensile stress** is defined, for any state of stress, σ_{ij}, in terms of the components of the corresponding stress deviator tensor by

$$\bar{\sigma} \equiv \sqrt{\frac{3}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}^{(\text{dev})}\sigma_{ij}^{(\text{dev})}} \geq 0$$

The yield condition for general multiaxial states of stress can be expressed as

$$\bar{\sigma} \leq \sigma_y$$

Is our general criterion for multiaxial yielding consistent with our previously-established uniaxial yield criterion $|\sigma| = \sigma_y$?

Uniaxial stress:
$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Stress deviator:
$$\begin{bmatrix} \sigma_{11}^{(\mathrm{dev})} & \sigma_{12}^{(\mathrm{dev})} & \sigma_{13}^{(\mathrm{dev})} \\ \sigma_{21}^{(\mathrm{dev})} & \sigma_{22}^{(\mathrm{dev})} & \sigma_{23}^{(\mathrm{dev})} \\ \sigma_{31}^{(\mathrm{dev})} & \sigma_{32}^{(\mathrm{dev})} & \sigma_{33}^{(\mathrm{dev})} \end{bmatrix} = \begin{bmatrix} \frac{2\sigma}{3} & 0 & 0 \\ 0 & \frac{-\sigma}{3} & 0 \\ 0 & 0 & \frac{-\sigma}{3} \end{bmatrix}$$

Mises stress measure:
$$\begin{aligned} \bar{\sigma} &= \sqrt{\frac{3}{2}\left\{(\frac{2\sigma}{3})^2 + (\frac{-\sigma}{3})^2 + (\frac{-\sigma}{3})^2\right\}} \\ &= |\sigma|\sqrt{\frac{3}{2}\left\{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right\}} \\ &= |\sigma| \end{aligned}$$

Mises yield specializes
to the uniaxial yield
Condition under uniaxial stress

$$\bar{\sigma} = \sigma_y \Leftrightarrow |\sigma| = \sigma_y$$

Equivalent Expressions for Mises Equivalent Tensile Stress

In terms of stress deviator components:

$$\bar{\sigma} \equiv \sqrt{\frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}^{(\mathrm{dev})} \sigma_{ij}^{(\mathrm{dev})}} \geq 0$$

In terms of stress components:

$$\bar{\sigma} = \sqrt{\begin{array}{l} \frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right] \\ +3\left[\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right] \end{array}}$$

In terms of principal stress values:

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]}$$

EXAMPLE: Combined tension and torsion of a thin-walled tube:

Stress components and relation to loads and tube geometry:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{\theta z} \\ 0 & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{zz} \doteq \frac{F}{2\pi \bar{R} t} \equiv \text{``}\sigma\text{''}; \qquad \sigma_{\theta z} \doteq \frac{M_t}{2\pi \bar{R}^2 t} \equiv \text{``}\tau\text{''}$$

Stress deviator components:

$$\left[\sigma_{ij}^{(\text{dev})}\right] = \begin{bmatrix} \frac{-\sigma}{3} & 0 & 0 \\ 0 & \frac{-\sigma}{3} & \tau \\ 0 & \tau & \frac{2\sigma}{3} \end{bmatrix}$$

Evaluate Mises stress and compare to Uniaxial yield strength

$$\bar{\sigma}^2 = \sigma^2 + 3\tau^2 \leq \sigma_y^2$$

The Mises yield condition for this stress state can be represented as an ellipse in a 2D space whose axes are ``σ'' and ``τ''

EXAMPLE (continued)

A tube of wall thickness t = 3 mm and mean radius $\bar{R}$ = 30 mm is made of a material having tensile yield strength σ_y = 500 MPa and is preloaded to an axial force F = 200 kN

What is the maximum torque that can be applied without causing yield in the tube?

rearrange Mises yield:

$$3\tau^2 \leq \sigma_y^2 - \sigma^2$$

load/stress/geometry:

$$3\left(\frac{M_t}{2\pi\bar{R}^2 t}\right)^2 \leq \sigma_y^2 - \left(\frac{F}{2\pi\bar{R}t}\right)^2$$

algebra…

$$|M_t| \leq \frac{2\pi\bar{R}^2 t}{\sqrt{3}}\sigma_y\sqrt{1-\left(\frac{F}{2\pi\bar{R}t\sigma_y}\right)^2}$$

numerical values & un

$$|M_t| \leq \frac{2\pi(30mm)^2 \times 3mm}{\sqrt{3}}\frac{500N}{mm^2}\sqrt{1-\left(\frac{2\times 10^5 N}{2\pi 30mm \times 3mm \times \frac{500N}{mm^2}}\right)^2}$$

ANSWER:

$$\leq 3.46\,kNm$$

EXAMPLE

A tube of axial length L= 200 mm, wall thickness t = 3 mm
and mean radius R = 30 mm
is made of a material having tensile yield strength σ_y = 500 MPa
and is preloaded to an axial force F = 200 kN The torque is increased
to its initial yield value (previously-determined value: M_t =3.46 kNm) with F
held constant. Then, with dF=0, the torque is further incremented by
an amount dM_t = 0.1 kNm.

The Young's modulus is E=208 GPa, ν=0.3, and the initial value of the plastic hardenig moduus is h = 3 GPa

QUESTION:

- Evaluate the stress increment, d σ_{ij}
- Evaluate the strain increment, d ε_{ij}
- Evaluate the increment in length of the tube, d L
- Evaluate the increment in the end-to-end rotation of the
- Two ends of the tube, d ϕ

Stress increment:

$$d\sigma_{ij} = 0, \quad \underline{\text{except}}$$

$$d\sigma_{z\theta} = d\sigma_{\theta z} = \frac{dM_t}{2\pi \bar{R}^2 t} = \frac{0.1 kNm}{2\pi (30\, mm)^2\, 3mm} \times \frac{10^3 mm}{m} = \frac{5.9\, N}{(mm)^2} = 5.9\, MPa$$

Matrix form:

$$\left[d\sigma_{ij} \right] = \begin{bmatrix} d\sigma_{rr} & d\sigma_{r\theta} & d\sigma_{rz} \\ d\sigma_{\theta r} & d\sigma_{\theta\theta} & d\sigma_{\theta z} \\ d\sigma_{zr} & d\sigma_{z\theta} & d\sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & d\sigma_{\theta z} \\ 0 & d\sigma_{z\theta} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5.9 \\ 0 & 5.9 & 0 \end{bmatrix} MPa$$

$$d\sigma_{\theta z} \doteq \frac{dM_t}{2\pi \bar{R}^2 t} \equiv d\tau$$

Strain increment:

$$
\begin{aligned}
d\epsilon_{ij} &= d\epsilon_{ij}^{(e)} + d\epsilon_{ij}^{(p)} \\
&= \underbrace{\frac{1}{E}\left[(1+\nu)\, d\sigma_{ij} - \nu\, \delta_{ij} \left(\sum_{k=1}^{3} d\sigma_{kk}\right)\right]}_{d\epsilon_{ij}^{(e)}} + \underbrace{\frac{3}{2}\, d\overline{\epsilon}^{(p)}\, \frac{\sigma_{ij}^{(\text{dev})}}{\bar{\sigma}}}_{d\epsilon_{ij}^{(p)}}
\end{aligned}
$$

Equivalent plastic strain increment:

$$
d\overline{\epsilon}^{(p)} \equiv \overbrace{\frac{ds}{h} = \frac{d\bar{\sigma}}{h}}^{\text{consistency:}\, ds = d\bar{\sigma}}
$$

Increment in Mises equivalent stress:

Formal definition:

$$
d\bar{\sigma} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial \bar{\sigma}}{\partial \sigma_{ij}}\, d\sigma_{ij}
$$

alternate derivation:

$$
\begin{aligned}
\bar{\sigma}^2 &= \frac{3}{2} \sum_{m=1}^{3} \sum_{n=1}^{3} \sigma_{mn}^{(\text{dev})} \sigma_{mn}^{(\text{dev})} \Rightarrow \\
2\,\bar{\sigma}\, d\bar{\sigma} &= 2\,\frac{3}{2} \sum_{m=1}^{3} \sum_{n=1}^{3} \sigma_{mn}^{(\text{dev})}\, d\sigma_{mn}^{(\text{dev})} \\
d\bar{\sigma} &= \frac{3}{2} \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{\sigma_{mn}^{(\text{dev})}}{\bar{\sigma}}\, d\sigma_{mn}
\end{aligned}
$$

Strain increment:

$$d\epsilon_{ij} = d\epsilon_{ij}^{(e)} + d\epsilon_{ij}^{(p)} = \underbrace{\frac{1}{E}\left[(1+\nu)\,d\sigma_{ij} - \nu\,\delta_{ij}\left(\sum_{k=1}^{3} d\sigma_{kk}\right)\right]}_{d\epsilon_{ij}^{(e)}} + \underbrace{\frac{3}{2}\,d\bar{\epsilon}^{(p)}\,\frac{\sigma_{ij}^{(\mathrm{dev})}}{\bar{\sigma}}}_{d\epsilon_{ij}^{(p)}}$$

Axial strain increment:

$$\begin{aligned}
d\epsilon_{zz} &= d\epsilon_{zz}^{(e)} + d\epsilon_{zz}^{(p)} \\
&= \underbrace{0}_{d\epsilon_{zz}^{(e)}} + \underbrace{\frac{3}{2}\frac{\sigma_{zz}^{(\mathrm{dev})}}{\bar{\sigma}}\frac{d\bar{\sigma}}{h}}_{d\epsilon_{zz}^{(p)}} \\
&= \underbrace{\frac{3\times 235\,MPa}{2\times 500\,MPa}}_{3\sigma_{zz}^{(\mathrm{dev})}/2\bar{\sigma}}\ \underbrace{\frac{3}{2}\frac{2\times 204\,MPa\times d\tau}{500\,MPa\times 3GPa}}_{d\bar{\sigma}/h} \\
&= 10^{-3}\,0.864\times\frac{5.9\,MPa}{3\,MPa} \\
d\epsilon_{zz} &= 1.7\times 10^{-3} = d\epsilon_{zz}^{(p)}
\end{aligned}$$

Shear strain increment:

$$\begin{aligned}
d\epsilon_{z\theta} = d\epsilon_{\theta z} &= d\epsilon_{z\theta}^{(e)} + d\epsilon_{z\theta}^{(p)} \\
&= \underbrace{\frac{1+\nu}{E}\,d\sigma_{z\theta}}_{d\epsilon_{z\theta}^{(e)}} + \underbrace{\frac{3}{2}\frac{\sigma_{z\theta}^{(\mathrm{dev})}}{\bar{\sigma}}\frac{d\bar{\sigma}}{h}}_{d\epsilon_{z\theta}^{(p)}} \\
&= \frac{1.3\,d\tau}{208GPa} + \underbrace{\frac{3\times 204\,MPa}{2\times 500\,MPa}}_{3\sigma_{z\theta}/2\bar{\sigma}}\ \underbrace{\frac{3}{2}\frac{2\times 204\,MPa\times d\tau}{500\,MPa\times 3GPa}}_{d\bar{\sigma}/h} \\
&= 10^{-3}\left[\frac{1.3}{208} + \frac{.3745}{3}\right]\frac{5.9\,MPa}{MPa} \\
d\epsilon_{z\theta} &= [0.0369 + 0.736]\times 10^{-3} = 0.774\times 10^{-3}
\end{aligned}$$

Radial and hoop strain increments:

$$d\epsilon_{rr} = d\epsilon_{\theta\theta} = d\epsilon_{rr}^{(p)} = d\epsilon_{\theta\theta}^{(p)} = -\frac{1}{2}d\epsilon_{zz}^{(p)} = -.864\times 10^{-3}$$

Note: $d\varepsilon_{rr}^{(e)} = 0$; $\sigma_{rr}^{(dev)} = -\sigma_{zz}^{(dev)}/2$; etc.

Strain/displacement relations:

Axial tube elongation increment:

$$dL = L \times d\epsilon_{zz} = 200\,mm \times 1.7 \times 10^{-3} = 0.34\,mm$$

Tube end-to-end rotation increment:

$$\begin{aligned} d\phi &= \frac{L}{\bar{R}} \times 2d\epsilon_{z\theta} \\ &= \frac{30\,mm}{200\,mm} \times 0.77 \times 10^{-3} \\ &= 0.116 \times 10^{-3}\text{radians} = 6.6 \times 10^{-3}\,\text{degrees} \end{aligned}$$