Yielding Under Multi-axial Stress

2.002 Mechanics and Materials II
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Uniaxial tension/compression:
initial linear elastic response,
as axial stress, σ, is increased
up to the uniaxial "yield condition":

$$|\sigma| \leq \sigma_y$$

Suppose that, at some location
in a body made of the same material,
the state of stress is multi-axial, with
cartesian components σ_{ij};

QUESTION: Will plastic deformation occur under this state of stress?

Approach: we need to define a non-negative scalar, stress-valued function of [all] the stress components, such that it can consistently generalize the uniaxial yield criterion, $|\sigma| < \sigma_y$

Observation # 1: pressure insensitivity of uniaxial yielding

Suppose that a uniaxial test is performed under fixed superposed hydrostatic pressure, *p,* so the cartesian stress components are

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma - p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Plastic deformation is observed to commence when $|\sigma| = \sigma_y$, essentially independent of the value of *p*

This suggests that yielding is ~ independent of the mean normal stress given by $\Sigma = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$

Recall the <u>stress deviator tensor</u>, whose components are given by

$$\left[\sigma_{ij}^{(\mathrm{dev})}\right] \equiv \left[\sigma_{ij}\right] - \frac{1}{3}\left(\sum_{k=1}^{3}\sigma_{kk}\right)\left[\delta_{ij}\right]$$

Clearly, *the stress deviator tensor is independent of the mean normal stress*

The **<u>Mises equivalent tensile stress</u>** is defined, for any state of stress, σ_{ij}, in terms of the components of the corresponding stress deviator tensor by

$$\bar{\sigma} \equiv \sqrt{\frac{3}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}^{(\mathrm{dev})}\sigma_{ij}^{(\mathrm{dev})}} \geq 0$$

The yield condition for general multiaxial states of stress can be expressed as

$$\bar{\sigma} \leq \sigma_y$$

Is our general criterion for multiaxial yielding consistent with our previously-established uniaxial yield criterion $|\sigma| = \sigma_y$?

Uniaxial stress: $$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Stress deviator: $$\begin{bmatrix} \sigma_{11}^{(\mathrm{dev})} & \sigma_{12}^{(\mathrm{dev})} & \sigma_{13}^{(\mathrm{dev})} \\ \sigma_{21}^{(\mathrm{dev})} & \sigma_{22}^{(\mathrm{dev})} & \sigma_{23}^{(\mathrm{dev})} \\ \sigma_{31}^{(\mathrm{dev})} & \sigma_{32}^{(\mathrm{dev})} & \sigma_{33}^{(\mathrm{dev})} \end{bmatrix} = \begin{bmatrix} \frac{2\sigma}{3} & 0 & 0 \\ 0 & \frac{-\sigma}{3} & 0 \\ 0 & 0 & \frac{-\sigma}{3} \end{bmatrix}$$

Mises stress measure: $$\begin{aligned} \bar{\sigma} &= \sqrt{\frac{3}{2}\left\{(\frac{2\sigma}{3})^2 + (\frac{-\sigma}{3})^2 + (\frac{-\sigma}{3})^2\right\}} \\ &= |\sigma|\sqrt{\frac{3}{2}\left\{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right\}} \\ &= |\sigma| \end{aligned}$$

Mises yield specializes
to the uniaxial yield
Condition under uniaxial stress

$$\bar{\sigma} = \sigma_y \Leftrightarrow |\sigma| = \sigma_y$$

Equivalent Expressions for Mises Equivalent Tensile Stress

In terms of stress deviator components:

$$\bar{\sigma} \equiv \sqrt{\frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij}^{(\text{dev})} \sigma_{ij}^{(\text{dev})}} \geq 0$$

In terms of stress components:

$$\bar{\sigma} = \sqrt{\begin{array}{l} \frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] \\ +3\left[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right] \end{array}}$$

In terms of principal stress values:

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

EXAMPLE: Combined tension and torsion of a thin-walled tube:

Stress components and relation to loads and tube geometry:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{\theta z} \\ 0 & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{zz} \doteq \frac{F}{2\pi \bar{R} t} \equiv \text{``}\sigma\text{''}; \qquad \sigma_{\theta z} \doteq \frac{M_t}{2\pi \bar{R}^2 t} \equiv \text{``}\tau\text{''}$$

Stress deviator components:

$$\left[\sigma_{ij}^{(\mathrm{dev})}\right] = \begin{bmatrix} \frac{-\sigma}{3} & 0 & 0 \\ 0 & \frac{-\sigma}{3} & \tau \\ 0 & \tau & \frac{2\sigma}{3} \end{bmatrix}$$

Evaluate Mises stress and compare to Uniaxial yield strength

$$\bar{\sigma}^2 = \sigma^2 + 3\tau^2 \leq \sigma_y^2$$

The Mises yield condition for this stress state can be represented as an ellipse in a 2D space whose axes are ``σ'' and ``τ''

EXAMPLE (continued)

A tube of wall thickness t = 3 mm and mean radius $\bar{R}$ = 30 mm is made of a material having tensile yield strength σ_y = 500 MPa and is preloaded to an axial force F = 200 kN

What is the maximum torque that can be applied without causing yield in the tube?

rearrange Mises yield: $$3\tau^2 \le \sigma_y^2 - \sigma^2$$

load/stress/geometry: $$3\left(\frac{M_t}{2\pi\bar{R}^2 t}\right)^2 \le \sigma_y^2 - \left(\frac{F}{2\pi\bar{R}t}\right)^2$$

algebra… $$|M_t| \le \frac{2\pi\bar{R}^2 t}{\sqrt{3}}\sigma_y\sqrt{1-\left(\frac{F}{2\pi\bar{R}t\sigma_y}\right)^2}$$

numerical values & un $$|M_t| \le \frac{2\pi(30mm)^2 \times 3mm}{\sqrt{3}}\frac{500N}{mm^2}\sqrt{1-\left(\frac{2\times 10^5 N}{2\pi 30mm \times 3mm \times \frac{500N}{mm^2}}\right)^2}$$

ANSWER: $$\le 3.46\,kNm$$