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2.002 Mechanics and Materials II  
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Laboratory Module No. 2  
Rate- and Time-Dependent Polymer Deformation

## 1 Objectives

In the mechanical testing performed to date, little explicit attention has focused on real-time histories of loading: tension and bending deformation have been imposed on a “laboratory time-scale”, and we have implicitly assumed that, for monotonic loading, the stress/strain response of the material under investigation does not depend strongly on the “time” parameter. For example, we have not considered the possibility that strain might continue to increase with time during extended periods of constant applied stress (“creep”), or that a structure subjected to a rapidly-applied imposed *displacement* which is then held constant in time might generate high initial loads (short-time), but show steady decay of load level with time (“relaxation”) over longer times. In this lab module, we will explore both of these types of material behavior for small deformations in various engineering polymers; the collective term describing such mechanical behavior is linear viscoelasticity.

## 2 Lab Tasks

In this laboratory module you will

1. Briefly be introduced to the phenomena of creep and relaxation in viscoelastic polymers.
2. Record data on room-temperature load relaxation in beam specimens of an engineering polymer subjected to an imposed mid-span bending displacement, and also to a beam specimen subjected to a history of imposed mid-span loading and unloading. The data will begin the quantification of *linear viscoelastic* behavior in the polymer, in particular, for low density polyethylene (LDPE).

3. Observe the frequency dependence of cyclic deformation in a viscoelastic polymer.

### 3 Laboratory Assignment: Specific Questions to Answer

1. Plot the mid-span deflection vs. time curve for the rapidly-applied mid-span loading applied to the LDPE beams, and use the correspondence principle to extract the creep function  $J_c(t)$ . Is the response immediately following the removal of load consistent with the superposition principle? Discuss.
2. Plot the load vs. time curve for the LDPE beam subjected to a suddenly-applied mid-span displacement, and use the correspondence principle to extract an estimate of the relaxation function  $E_r(t)$ .
3. How closely do the creep and relaxations functions come to satisfying the reciprocal relation  $E_r(t) \times J_c(t) = 1$ ? Is this satisfied better for short times? Discuss.
4. Discuss how the relaxation data might be used to fit a linear viscoelastic three-element model (standard linear solid) to the time-dependent relaxation modulus,  $E_r(t)$ .
5. Plot the steady hysteresis curves (elliptical loops of cyclic stress vs. strain) at different frequencies,  $\omega$ . Does the phase lag  $\delta(\omega)$  exhibit expected trends? Discuss.

## 4 A Brief Background on Engineering Polymers

### 4.1 Springs and Dashpots

When a suddenly-applied tensile stress of magnitude  $\sigma$  is applied (say, at time  $t = 0$ ), and then held constant with time, the resulting history of strain in a linear elastic solid (often idealized by a “spring” element) is simple: the strain  $\epsilon(t) = 0$  (if  $t < 0$  corresponds to a time before application of stress), while  $\epsilon(t) = \sigma/E$  for times  $t > 0$ . Here  $E$  is the constant elastic tensile modulus of the material; evidently, the long-term strain is simply constant.

Conversely, when stress is suddenly applied to a linearly viscous fluid (as often idealized by a “dashpot” element), the fluid undergoes time-dependent straining. The constitutive model of a linear viscous material can be idealized by the relation

$$\dot{\epsilon}(t) \equiv \frac{d\epsilon(t)}{dt} = \frac{1}{\eta} \sigma(t),$$

where the *viscosity*  $\eta$  (dimensions:  $(t \times (F/L^2))$ ; typical units:  $(\text{sec} \cdot \text{MPa})$ ) is a constant for each fluid (and temperature!). Under the rapid stress-jump history previously discussed, the resulting strain history in a viscous fluid is

$$\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ (t/\eta) \sigma & \text{if } t \geq 0 \end{cases}$$

Thus, while stress is maintained, linear viscous fluids continue to deform.

It has been found, especially for polymeric materials, both engineered and natural, that the mechanical response under modest levels of stress contains elements of both the “linear elastic” and “linear viscous” idealizations noted above. Accordingly, the resulting behavior has been termed “**linear viscoelastic**”.

### 4.2 Polymer Structures and Characterization

Engineering polymers represent a broad class of engineering materials. They can be broadly categorized by features of the structures into which their long polymeric chain molecules are processed in the solid state. When the chains are randomly intertwined, the local structure is termed **amorphous**. If segments of polymer chains can locally come together in regular aligned arrays, those domains possess a sufficient degree of crystalline order to scatter light, x-rays, etc., and are hence termed crystalline regions. In general, the topological constraints on long polymer chains (think of a bowl of spaghetti!) prevent the chains from organizing completely into crystalline domains, so even when crystal organization is possible, polymers contain a substantial volume

fraction of amorphous, or disordered, material; thus, these polymers are termed **semi-crystalline**.

A second major organizing idea for categorizing amorphous polymers concerns the predominant type of bonding between chain segments. Relatively weak hydrogen bonds and van der Waals bonds permit easy relative motion of the disordered collection of chains, especially at higher temperatures. At sufficiently high temperatures, such polymers are viscous liquids, and they can be processed in the fluid state. Such amorphous polymers also have a characteristic **glass transition temperature**,  $T_g$ , near which their stiffness changes dramatically. Within a 10 to 20°C interval about  $T_g$ , the elastic stiffness can change by a factor of  $\sim 1000$ , with low-temperature, short-term elastic modulus of order  $\sim 1$  GPa well below  $T_g$ , and dropping to only  $\sim 1$  MPa a few degrees above  $T_g$ . Since these polymers can be fluid-processed into molds at high temperatures, then cooled to stiff solid engineering materials, they are termed **thermoplastic** polymers. Both amorphous and semi-crystalline polymers can be thermoplastic, with the understanding that for the semi-crystalline polymers, the fluid processing must take place at a temperature above  $T_m$ , the melting temperature of the crystallites; otherwise, the presence of the crystallites in a surrounding amorphous matrix would dramatically increase the viscosity to values too high for economic fluid-phase processing.

Amorphous polymers do not scatter visible light; they are typically transparent, and examples of amorphous thermoplastic polymers include polycarbonate (PC), polymethyl methacrylate (PMMA), and polystyrene (PS). Such polymers are typically used at temperatures well below their respective  $T_g$  values ( $T_g$  is near 100°C for PS and PMMA; for PC it is higher, near 150°). Collectively, these are also sometimes termed **glassy polymers**, with the understanding that they are used at temperatures well below  $T_g$ , where their relatively high elastic moduli (of order GPa) is characteristic of the glass-like behavior.

The crystallites of semi-crystalline polymers typically do scatter visible light; thus, they are generally opaque. Examples of important engineering semi-crystalline thermoplastic polymers are low-density polyethylene (LDPE), high-density polyethylene (HDPE), polypropylene (PP), polyamide (PA) (duPont trade name: “nylon”), polytetrafluoroethylene (PTFE) (duPont trade name: “teflon”), and poly-vinyl chloride (PVC). Each of these materials has some crystalline volume fraction,  $f_c = V_c/V_{tot}$ , and a remaining amorphous volume fraction,  $f_a = 1 - f_c$ . In HDPE,  $f_c \doteq 0.90$ , while in PVC,  $f_c$  is only  $\sim 0.10$ ; PA, LDPE, and PP typically have  $0.65 \leq f_c \leq 0.75$ . Recall, however, that the remaining amorphous volume fractions of each semicrystalline polymer must have its own glass transition temperature,  $T_g$ , and the operating temperature,  $T$ , of the amorphous/crystalline “composite” will greatly influence the stiffness of the amorphous domains, and thus, albeit to a lesser extent, of the overall polymeric material. For the amorphous portions of LDPE and HDPE, the respective  $T_g$  values are near  $-90^\circ\text{C}$  and  $-100^\circ\text{C}$ , while their crystalline melting temperatures  $T_m$  are near  $115^\circ\text{C}$  and  $135^\circ\text{C}$ , respectively. In contrast, for PA66,  $T_g \doteq 57^\circ$  and

$T_m = 265^\circ C$ . Thus, the semi-crystalline polyethylenes operate at room temperature ( $\sim 23^\circ C$ ) with their amorphous regions far into the viscous region, even though their larger volume fractions of crystalline order are well below their melting temperatures.

For completeness, we note that another type of inter-chain bonding occurs in amorphous polymers: strong primary “cross-linking” bonds can be introduced via *in situ* chemical reactions: such amorphous polymers are called **thermosets**. Important examples include epoxies. Once the chemical cross-linking has been performed, thermosets can no longer become viscous flowing fluids, even at very high temperatures: the primary bonds keep the chains from flowing past each other. Instead, the thermosets char at sufficiently elevated temperatures.

With this very brief background into the molecular structures of engineering polymers, we turn again to their mechanical behavior. From the discussion, we can expect that predominantly linear elastic response should be obtained from the crystalline regions and from the amorphous regions, providing the latter are well below  $T_g$ . However, near and, especially, above  $T_g$ , the amorphous domains will exhibit viscous flow-type behavior that should be evident in the macroscopic time-dependent stress-strain relations of the polymer: one part of that behavior, generally valid for total strains less than, say, 0.01, is termed **linear viscoelasticity**.

## 5 A Brief Introduction to Linear Viscoelasticity

(See also: Dowling, Sections 5.2.2-5.2.4; 15.6.1; 15.10  
Crandall, Dahl, and Lardner, section 5.18)

### 5.1 Stress Relaxation and Creep Compliance

The fundamental linear viscoelastic response of a material can be experimentally obtained from 2 related, but distinct, types of tests. In a **stress relaxation** test, a (small) strain of magnitude  $\epsilon_0$  is suddenly applied at time  $t = 0$ ; thus the history of imposed strain,  $\epsilon(t)$ , is

$$\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ \epsilon_0 & \text{if } t \geq 0. \end{cases}$$

The measured response to this imposed strain history is the stress history,  $\sigma(t)$ . In general,  $\sigma(t) = 0$  if  $t < 0$ , but will be non-zero and changing for times  $t > 0$ . Basic features of such a history are that the largest values of stress are recorded just after applying the strain (at  $t \rightarrow 0^+$ ), with a smooth, steady decay to lower values for longer times. We can extract the material **stress relaxation function**,  $E_r(t)$ , from

the data by defining

$$E_r(t) \equiv \frac{\sigma(t)}{\epsilon_0}.$$

The short-term value of the function is termed its ‘glassy’ response,  $E_{rg} \equiv E_r(t \rightarrow 0^+)$ , while its long-term value is termed the ‘equilibrium’ value:  $E_{re} \equiv E_r(t \rightarrow \infty)$ .

The second fundamental testing mode, called a creep test, consists of suddenly imposing a (small) constant stress, of magnitude  $\sigma_0$ , at time  $t = 0$ . The resulting strain history,  $\epsilon(t)$ , is recorded for times  $t \geq 0$ . We define the **creep function** by

$$J_c(t) \equiv \frac{\epsilon(t)}{\sigma_0}.$$

Major features of this function are a small, ‘glassy’ value just after loading:  $J_{cg} \equiv J_c(t \rightarrow 0^+)$ , and a steadily increasing value toward a long-time ‘equilibrium’ value,  $J_{ce} \equiv J_c(t \rightarrow \infty)$ .

Viscoelastic behavior is said to be **linear** when the value of  $E_r(t)$  is independent of the value chosen for  $\epsilon_0$  (similarly, when  $J_c(t)$  is independent of the magnitude of  $\sigma_0$ ). Thus, for example, if the response to a suddenly-imposed strain of magnitude  $\epsilon_0$  is  $\sigma(t) = E_r(t)(\epsilon_0)$ , then, for **linear** viscoelasticity, the response to a suddenly-applied strain of magnitude  $\alpha \times \epsilon_0$  would be  $\alpha \times \sigma(t) = E_r(t)(\alpha \times \epsilon_0)$ . Although it is generally found that

$$J_{cg} = 1/E_{rg} \quad \text{and} \quad J_{ce} = 1/E_{re},$$

in general, for intermediate times  $t$ ,

$$J_c(t) \neq 1/E_r(t),$$

although in practice, the numerical difference between the two is seldom more than a few percent, so the two responses are sometimes used as if they were truly reciprocal in the time domain.

## 5.2 Superposition Principle

An important consequence of linearity of viscoelastic response is **superposition**. Suppose that an unstressed solid is subjected to a suddenly-applied stress jump, of magnitude  $\Delta\sigma_1$ , at time  $t_1 = 0$ , and stress is held constant at that value until the time  $t = t_2 > t_1$ . At time  $t_2$ , another suddenly-applied stress change, of magnitude  $\Delta\sigma_2$  occurs, and for times  $t > t_2$ , stress is again held constant, but at the value  $\Delta\sigma_1 + \Delta\sigma_2$ . In this case, the resulting strain history will be

$$\epsilon(t) = \Delta\sigma_1 J_c(t - t_1) + \Delta\sigma_2 J_c(t - t_2).$$

Recall that the creep function is zero-valued when its argument is negative: thus, for times  $t$  less than  $t_1$ , the argument  $t - t_1 < 0$ , and no strain occurs. Similarly, the only

thing that matters for any given stress jump is the (positive) time interval since the jump was applied. Thus, the relevant time argument for response to the first stress jump is  $t - t_1$ , and so on. The mathematical structure of linear viscoelasticity is made particularly evident when the special choice of  $\Delta\sigma_2 = -\Delta\sigma_1$  is made, corresponding physically to sudden application of stress at time  $t_1$ , followed by a subsequent rapid removal of that stress at time  $t_2$ . (See Dowling Figure 5-5 for a graphical schematic of the resulting response.)

### 5.3 Analogue Models of Linear Viscoelastic Response

We can get a physical feel for linear elastic behavior by constructing simple analogue models constructed from assemblies of linear elastic spring elements and constant-viscosity mechanical dashpots. While the parameters of such models (individual spring stiffnesses and dashpot viscosities) can be adjusted to best match experimental data, we emphasize that the agreement will be only qualitative at best, since the time-dependent behavior of the material itself is generally considerably more complex than can be represented by a simple analogue model. Nonetheless, we introduce a particular such model, termed the **standard linear solid**, in this section.

This model consists of two springs and a dashpot, connected as shown in figure 1. One branch of the model consists of a single linear elastic spring. A second branch, loaded in parallel with the first branch, consists of a second linear elastic spring element in series with a linear viscous dashpot (such a series branch is termed a **Maxwell model**). Thus, the time-dependent total stress  $\sigma(t)$  carried by the two parallel-loaded branches is

$$\sigma(t) = \sigma_1(t) + \sigma_2(t),$$

where  $\sigma_1(t)$  is the stress in the elastic branch, and  $\sigma_2(t)$  is the stress carried by both members of the second (Maxwell) branch.

The notion is that the total “strain” in the system,  $\epsilon(t)$ , is applied to both branch number one (whose strain equals  $\epsilon_1(t)$ ), and to the second “branch”, whose strain is  $\epsilon_2(t)$ :

$$\epsilon(t) = \epsilon_1(t) = \epsilon_2(t).$$

Moreover, within the series spring/dashpot branch number two, the strain is the sum of the elastic strain in the spring ( $\sigma_2/E_2$ ) and the viscous strain in the dashpot:

$$\begin{aligned} \epsilon_2(t) &= \epsilon_{\text{spring2}}(t) + \epsilon_{\text{viscous}}(t) \\ &= \frac{\sigma_2(t)}{E_2} + \epsilon_{\text{viscous}}(t). \end{aligned}$$

Since the stress on the dashpot is also that within spring 2 ( $\sigma_2(t)$ ), we can as well

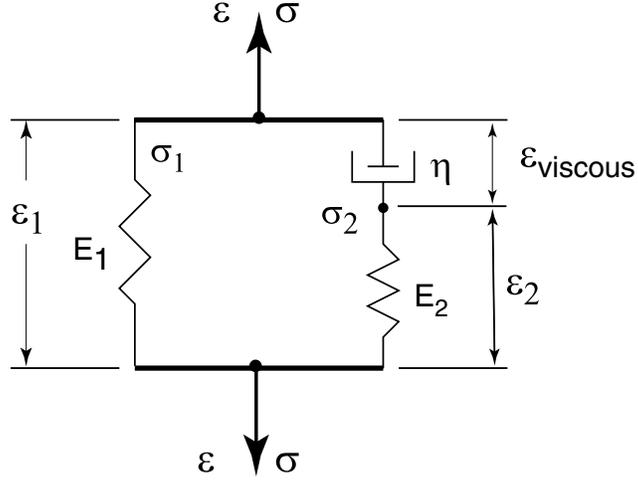


Figure 1: Three-element spring/dashpot model of linear viscoelastic behavior: the standard linear solid. **NOTE: the figure, as drawn, is slightly inconsistent with the narrative in that the figure indicates that “ $\epsilon_2$ ” is the (elastic) strain in spring number 2, while the narrative uses the term “ $\epsilon_2(t)$ ” to denote the sum of the time-dependent viscous strain and the elastic strain in spring 2. Please follow the narrative in relation to equations developed.**

sum their strain rates by differentiating the previous equation with respect to time:

$$\frac{d\epsilon_2}{dt} \equiv \dot{\epsilon}_2(t) = \frac{1}{E_2} \dot{\sigma}_2(t) + \frac{1}{\eta} \sigma_2(t).$$

Here we have made use of the viscous dashpot constitutive model  $\dot{\epsilon}_{\text{viscous}}(t) = \eta \sigma_2(t)$ . The stress in spring 1 ( $\sigma_1(t)$ ) drives linear elastic response in spring 1, resulting in  $\epsilon_1(t) = \sigma_1(t)/E_1$ , an expression which can also be phrased in terms of strain rates and stress rates as  $\dot{\epsilon}_1(t) = \dot{\sigma}_1(t)/E_1$ .

Compatibility of the two branches of the assembly requires, in rate form,

$$\dot{\epsilon}(t) = \dot{\epsilon}_1(t) = \dot{\epsilon}_2(t),$$

or

$$\dot{\epsilon}(t) = \frac{1}{E_1} \dot{\sigma}_1(t) = \frac{1}{E_2} \dot{\sigma}_2(t) + \frac{1}{\eta} \sigma_2(t).$$

Finally, note that the total stress,  $\sigma(t)$ , is the sum of the stresses carried by the two members in parallel:

$$\sigma(t) = \sigma_1(t) + \sigma_2(t),$$

and in rate form

$$\dot{\sigma}(t) = \dot{\sigma}_1(t) + \dot{\sigma}_2(t).$$

These equations can be re-arranged substituted in to eliminate the “internal” stress values in favor of model constants and total stress and strain:

$$\sigma_2(t) = \sigma(t) - E_1 \epsilon(t); \quad \dot{\sigma}_2(t) = \dot{\sigma}(t) - E_1 \dot{\epsilon}(t).$$

Now consider the response of the 3-element model to the step-change in total strain, of magnitude  $\epsilon_0$ , applied at time  $t = 0$ . Again, prior to application of the stress ( $t < 0$ ), there is no stress or strain in any element, and  $\sigma(t) = 0$ . At times  $t > 0$ , the strain in spring 1 is simply  $\epsilon_1(t) = \epsilon_0$ , so the stress in spring 1 is constant, of magnitude  $\sigma_1(t) = E_1 \epsilon_0$ . The total series strain of spring 2 and the dashpot is also constant, of magnitude  $\epsilon_0$ , so that  $\dot{\epsilon}_{\text{series}}(t) = 0$ . Referring to the series strain-rate equation above, we are thus lead to a differential equation describing  $\sigma_2(t)$  for times  $t > 0$ :

$$\begin{aligned} \dot{\sigma}_2(t) + \frac{E_2}{\eta} \sigma_2(t) &= 0 \Rightarrow \\ \dot{\sigma}_2(t) &= -\frac{E_2}{\eta} \sigma_2(t). \end{aligned}$$

This simple first-order ordinary differential equation leads to the decaying exponential solution form

$$\sigma_2(t) = \sigma_2(t = 0^+) \exp^{-(E_2/\eta)t},$$

where  $\sigma_2(t = 0^+)$  is the stress in the series member at time  $t = 0^+$ , just after sudden application of the total strain jump. A little reflection will lead to the insight that, since stress remains finite during the vanishingly small time interval of loading, the viscous dashpot contributes zero strain in the limit as  $t \rightarrow 0^+$ ; thus, at this short-time limit, the total strain in the series element is completely accommodated by spring 2:  $\epsilon_{\text{spring2}}(t = 0^+) = \epsilon_0$  and  $\epsilon_{\text{viscous}}(t = 0^+) = 0$ . Thus, using the elastic constitutive relation for spring 2 at time  $t = 0^+$ ,

$$\sigma_2(t = 0^+) = E_2 \epsilon_{\text{spring2}}(t = 0^+) = E_2 \epsilon_0.$$

On substituting this result into the decaying exponential form for  $\sigma_2(t)$ , there results

$$\sigma_2(t) = E_2 \exp^{-(E_2/\eta)t} \epsilon_0.$$

Finally, we can add the two stress contributions,  $\sigma_1(t)$  and  $\sigma_2(t)$ , to give the total stress for this loading of the 3-element model as

$$\sigma(t) = E_1 \epsilon_0 + E_2 \exp^{-(E_2/\eta)t} \epsilon_0 \equiv E_r(t) \epsilon_0,$$

where the relaxation modulus function  $E_r(t) = E_1 + E_2 \exp^{-(E_2/\eta)t}$  has short term value  $E_r(t = 0^+) \equiv E_0 = E_1 + E_2$ , and long-term value  $E_r(t \rightarrow \infty) \equiv E_\infty = E_1$ . Thus,

$E_2 = E_0 - E_\infty$ . Moreover, the system relaxes with a characteristic time constant “ $\tau$ ”, where  $\tau \equiv \eta/E_2$ . Using these new variables, we can express the relaxation modulus of the model standard linear solid (SLS) system as

$$E_{r(\text{SLS})}(t) = E_\infty + (E_0 - E_\infty) \exp^{-t/\tau}.$$

In this form, the 3 independent material properties of the model are long- and short-term elastic moduli, and characteristic relaxation time.

Evidently,  $E_0 = E_{rg}$  and  $E_\infty = E_{re}$ .

A similar analytical treatment can be applied to extract the creep response of the standard linear solid when subjected to a suddenly-applied total stress, although the algebra is a bit tedious.

## 5.4 Correspondence Principle

Within linear viscoelasticity, we can make use of the **correspondence principle** to apply (known!) solutions for *linear elastic* structures in construction of the time-dependent solutions for geometrically identical bodies made of *linear viscoelastic* materials. In particular, suppose that a simply-supported beam of length  $L$  is subject to a sudden application of mid-span displacement of magnitude  $\delta$ ; had the beam been made of linear elastic material, we are familiar with the relation between mid-span load,  $P$ , and mid-span deflection,  $\delta$ , as

$$\delta = \frac{PL^3}{48EI},$$

where  $I$  is the area moment of inertia of the beam cross-section, and  $E$  is the (constant) elastic modulus. We can re-arrange this equation to express the (constant) load in terms of the displacement as

$$P = \frac{48EI}{L^3} \delta.$$

But now consider the same application of sudden mid-span displacement of magnitude  $\delta$  to a linear viscoelastic beam. We can simply replace “ $E$ ” of the linear elastic solution with  $E_r(t)$  in order to calculate the time-dependent mid-span load,  $P(t)$ :

$$P(t) = \frac{48E_r(t)I}{L^3} \delta.$$

Among the other features of the correspondence principle, one may note that all of the strain components in the linear viscoelastic body have the same (constant) values

that they did in the corresponding linear elastic solution; e.g., for  $0 \leq x/L \leq 1/2$ ,

$$v_{\text{viscoelastic}}(x, t) = v_{\text{elastic}}(x) = \delta \left( \frac{x}{L} \right) (3 - 4(x/L)^2),$$

while local time-dependent bending moment distribution in the beam, again for  $0 \leq x/L \leq 1/2$ , is

$$M(x, t) = \frac{xP(t)}{2} = \frac{24xE_r(t)I}{L^3} \delta,$$

and the time- and space-dependent axial stress component in this end of the beam is

$$\sigma_{xx}(x, y; t) = -\frac{yM(x, t)}{I} = -24E_r(t) \left( \frac{xy}{L^2} \right) \left( \frac{\delta}{L} \right).$$

It would be tempting to extend the correspondence principle just illustrated to situations involving sudden application of stress (or load), (say, mid-span load suddenly increased to  $P_0$  and then held constant), and calculate the (now time-dependent!) mid-span deflection as

$$\delta(t) \stackrel{(?)}{=} \frac{L^3}{48IE_r(t)} P_0.$$

Regarding this enticing possibility, there is both “good news” and “bad news”. First the bad news: in general, the mathematics of linear viscoelasticity tells us that this formulation, where  $E_r(t)$  describes stress relaxation in response to a suddenly-applied *strain*, is not strictly applicable to the present case of a suddenly-applied *stress*. Now the good news: in practice, the “error” associated with doing this anyway (even though it is not rigorous) is typically small, so you can get good long-term and short-term answers, with the largest of the (rather small) errors occurring at intermediate times when the system is changing most rapidly.

With some reflection, you may surmise that the rigorous answer to the deflection of the beam with a suddenly-applied load comes from the creep function  $J_c(t)$ ; indeed, the time-dependent mid-span deflection is

$$\delta(t) = \frac{J_c(t)L^3}{48I} P_0,$$

and the stress and bending moment distributions are time-independent:

$$M(x, t) = \frac{P_0 x}{2};$$

$$\sigma_{xx}(x, y, t) = -\frac{yM(x, t)}{I} = -\frac{xyP_0}{2},$$

again in  $0 \leq x/L \leq 1/2$ .

## 6 Sinusoidal Loading of Viscoelastic Materials and Vibration Damping

Because of the complex interplay of viscous and elastic deformation in linear viscoelastic materials, they exhibit an important phenomenon, termed **damping**, that relates to viscous mechanical losses during cyclic loading. Damping characteristics are particularly important in limiting the amplitude of steady mechanical vibrations in structures subjected to periodic forcing, and in rapidly diminishing the amplitude of transient free vibrations.

Suppose that a linear viscoelastic material is subjected to a sinusoidal history of stressing,

$$\sigma(t) = \sigma_a \sin \omega t,$$

for a small stress amplitude,  $\sigma_a$ , and a given frequency of loading,  $\omega$  (radians/second). The steady-state strain response of the material, after many loading cycles, will settle to a periodic response of amplitude  $\epsilon_a$ , frequency  $\omega$ , but with a **phase shift** of magnitude “ $\delta$ ” (radians), of the form

$$\epsilon(t) = \epsilon_a \sin(\omega t - \delta).$$

(See Dowling, section 15.10).

The magnitude of the phase shift,  $\delta(\omega)$ , depends on  $\omega$ , the frequency of loading; in general,  $\delta$  is very small for both very fast cycling (large  $\omega$ ), in which mostly glassy elastic response occurs, and also for very slow cycling (very small  $\omega$ ), for which mostly equilibrium elastic response occurs. For intermediate cycling frequencies, however,  $\delta$  increases in magnitude; for a given temperature, typically a maximum value of phase shift ( $\delta_{\max}$ ) occurs at a particular frequency, which can be termed  $\omega_d$ . (See Dowling Fig. 15.35).

The magnitude of damping of vibrations is related to a measure  $Q^{-1} \equiv \tan \delta$ . Here  $Q$  is termed the **quality factor**, and is also related to the **log decrement**,  $\Delta_t$ , by

$$\Delta_t = \pi \tan \delta = \frac{\pi}{Q}.$$

The physical significance of the log decrement to free vibration is that the ratio of successive-cycle vibration amplitudes, with, say, vibration amplitude (e.g., tip displacement in a vibrating cantilever) “ $d_n$ ” for cycle “ $n$ ” and “ $d_{n+1}$ ” for cycle “ $n + 1$ ”, decays according to

$$\Delta_t = \ln \left( \frac{d_n}{d_{n+1}} \right).$$

Thus materials and structures with high values of  $\tan \delta$  (and thus small  $Q$ -values and large  $\Delta_t$ -values) strongly damp out vibrations. Conversely, when  $\delta$  is small, only light damping occurs.