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2.002 MECHANICS AND MATERIALS II

SOLUTION for QUIZ NO. 1

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Problem 2 (20 points)

- For an imposed strain history consisting of a rapidly applied step-function jump in strain of magnitude ϵ_0 , and applied at time $t = 0$, the stress relaxation function $E_r(t)$ is defined by

$$E_r(t) = \frac{\sigma(t)}{\epsilon_0}$$

The stress relaxation function acknowledges that in a linear viscoelastic material stress/strain is a function of time. In linear viscoelastic behavior, the time dependence will die out and the stress will eventually reach a "relaxed value".

- For an imposed stress history consisting of a rapidly applied step-function jump in stress of magnitude σ_0 , and applied at time $t = 0$, the creep function $J_c(t)$ is defined by

$$J_c(t) = \frac{\epsilon(t)}{\sigma_0}$$

The creep function is used when a constant stress is applied and the strain then changes with time (or "creeps".) In linear viscoelastic behavior, it will creep until the time dependence dies out and reaches a constant strain.)

- The Correspondence Principle states that linear elastic models can be applied to linear viscoelastic problems with equal geometries by replacing E with $E_r(t)$ or $1/J_c(t)$ depending on whether the displacement or load is held fixed in time.

Example: For a simply supported beam with a constant point load in the center, the displacement at the center is modelled by

$$\delta = \frac{-PL^3}{48EI}$$

If the beam is made of a linear-viscoelastic material and the load P is applied suddenly at time $t = 0$, the deflection will vary over time and E can be replaced by $\frac{1}{J_c(t)}$

$$\delta(t) = \frac{-PL^3 J_c(t)}{48I}$$

- The molecular structure of an amorphous polymer consists of long polymeric chains that are randomly oriented and intertwined. It is termed amorphous because there is no order to its structure. The solid is held together in two ways: through weak van der Waals and hydrogen bonding and also through the mechanical intertwining of the chains.

The major feature of amorphous polymers is the glass transition temperature (T_g). When the temperature of the polymer is greater than T_g , there is enough thermal energy in the chains of atoms to break free from most of the van der Waals bonds. Thus when an amorphous polymer is loaded, the chains move more freely than if the van der Waals bonds were present. Thus it takes less load to displace the material a certain amount when the material is above T_g compared to below and a very large drop in E is noticed when $T > T_g$. Also, E of amorphous polymers are less than the elastic modulus of crystalline and semi-crystalline polymers.

The time dependent mechanics arises through the intertwining of the chains and the weak bonds. Linear viscoelastic behavior is noticed even at low temperatures much below T_g , even though the behavior is mostly linear elastic and only slightly viscous. The opposite occurs at temperatures above T_g . For $T > T_g$ the material is mostly a very viscous fluid, and shows only slight linear elasticity. Regardless, the two behaviors are seen for most temperatures and the material is linear viscoelastic.

The polymer chains will begin to line up under plastic deformation when $T < T_g$, or under most loadings when $T > T_g$.

Problem 3 (40 points)

- In the central portion of the composite plate, the top and bottom surfaces of the plate are free boundaries, which have zero traction on it. For example, for the top surface, the traction is

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma} \mathbf{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix} = 0 \quad (1)$$

We know if the vector $\mathbf{t}^{(n)} = 0$, its components must be equal to zero. So, $\sigma_{13} = 0$, $\sigma_{23} = 0$, and $\sigma_{33} = 0$ on the top surface. Similarly, we know at the bottom surface $\sigma_{13} = 0$, $\sigma_{23} = 0$, and $\sigma_{33} = 0$.

Additionally, since the stress tensor is symmetric, we have $\sigma_{31} = \sigma_{13} = 0$, $\sigma_{32} = \sigma_{23} = 0$. The only components left are in the 1-2 plane, σ_{11} , σ_{22} , and $\sigma_{12} = \sigma_{21}$; this is plane

stress. By assuming plane stress in the middle layer as well, traction will be zero on both side of the interface, and therefore equal.

- At the bonding interface, since the two materials are bonded perfectly, the displacement must be continuous across the interface. At the bonding plane, for each material, we have $u_1^{(Al)} = u_1^{(PC)}$ and $u_2^{(Al)} = u_2^{(PC)}$. From the strain-displacement relations,

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}; \epsilon_{22} = \frac{\partial u_2}{\partial x_2}; \epsilon_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \quad (2)$$

we can get $\epsilon_{11}^{(Al)} = \epsilon_{11}^{(PC)}$, $\epsilon_{22}^{(Al)} = \epsilon_{22}^{(PC)}$, and $\epsilon_{12}^{(Al)} = \epsilon_{12}^{(PC)}$.

- As the temperature change ΔT is constant in all layers, and the total strain is also constant in all layers, from the constitutive equations,

$$\sigma_{ij} = \frac{E}{1+\nu} \left[\epsilon_{ij} + \frac{\nu}{1-2\nu} \left(\sum_{k=1}^3 \epsilon_{kk} \right) \delta_{ij} - \frac{1+\nu}{1-2\nu} \alpha \Delta T \delta_{ij} \right] \quad (3)$$

the stress components will be constant within respective layers, i.e. $\sigma_{ij} = \sigma_{ij}^{(Al)}$ in the outer layer, and $\sigma_{ij} = \sigma_{ij}^{(PC)}$ in the central layer, for to-be-determined sets of constants $\sigma_{ij}^{(Al)}$ and $\sigma_{ij}^{(PC)}$.

- As shown in the above, we know the stress will be constant throughout respective layers. Let us make a rectangular region cutting from the center till the edge that perpendicular to x_1 axis throughout the thickness. At the edge side, there is no force acting on it since it is zero traction boundary. So, at the interior boundary, the net reaction force should also be zero. We already knew that the plate is in plane stress state.

In the figure 1 , the plane cut perpendicular to x_2 will also have the zero net force.

Because of the material and geometric symmetry of the plate along the $x_1 - x_2$ plane cross the origin, the stress at the top Al layer and bottom Al layer will have same value.

$$\begin{aligned} 0 &= \int_{-t/2}^{t/2} \sigma_{11}(x_3) dx_3 \\ &= 2\sigma_{11}^{(Al)} A^{(Al)} + \sigma_{11}^{(PC)} A^{(PC)} \\ &= 2\sigma_{11}^{(Al)} L_2 t_{(Al)} + \sigma_{11}^{(PC)} L_2 t_{(PC)} \end{aligned} \quad (4)$$

$$0 = 2\sigma_{11}^{(Al)} t_{(Al)} + \sigma_{11}^{(PC)} t_{(PC)} \quad (5)$$

Following a similar argument along the axis x_2 , we can get the similar relation held for the x_2 component of force balance.

Actually, because of the material and geometric symmetry of the plate along the axis x_1 and x_2 , we can derive that $\sigma_{11} = \sigma_{22}$ in respective layers.

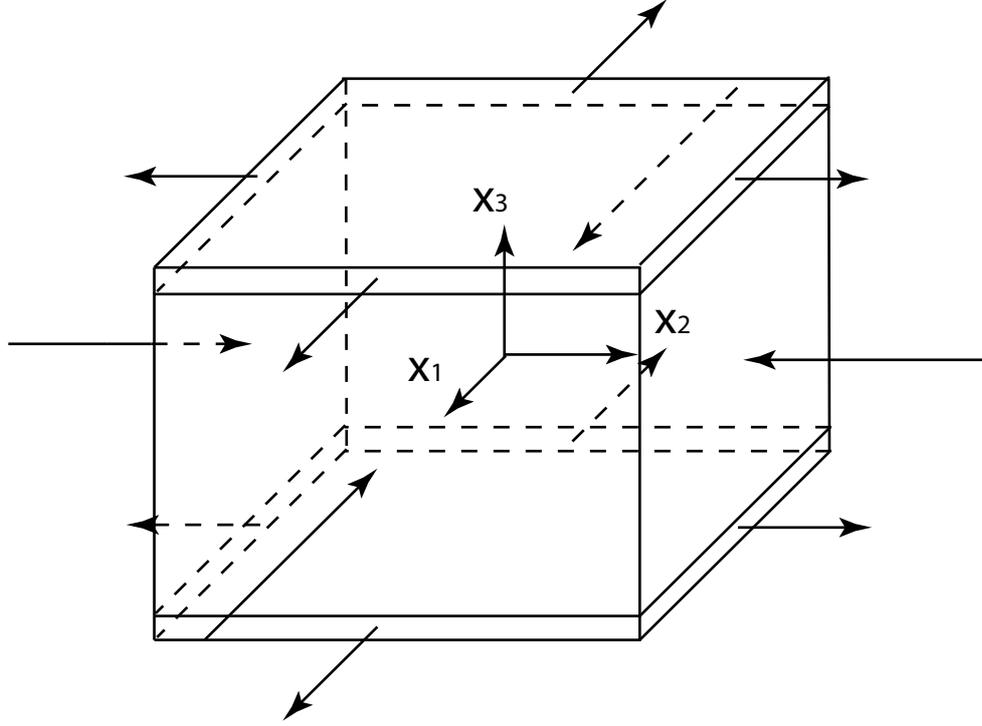


Figure 1: free body diagram of a section cut from the central portion

- The plate is in plane stress state. The only non-zero stress components are σ_{11}, σ_{22} and σ_{12} . According to the constitutive equation,

$$\epsilon_{ij} = \alpha \Delta T \delta_{ij} + \frac{1}{E} \left[(1 + \nu) \sigma_{ij} - \nu \delta_{ij} \left(\sum_{k=1}^3 \sigma_{kk} \right) \right] \quad (6)$$

we have

$$\epsilon_{11}^{(Al)} = \alpha_{Al} \Delta T + \frac{1}{E_{Al}} (\sigma_{11}^{(Al)} - \nu_{Al} \sigma_{22}^{(Al)}) = \epsilon_0 \quad (7)$$

As shown in the previous part, $\sigma_{11} = \sigma_{22}$, we have

$$\epsilon_{11}^{(Al)} = \alpha_{Al} \Delta T + \frac{1}{E_{Al}} \sigma_{11}^{(Al)} (1 - \nu_{Al}) = \epsilon_0 \quad (8)$$

Similarly, we have the constitutive relation in PC layer

$$\epsilon_{11}^{(PC)} = \alpha_{PC}\Delta T + \frac{1}{E_{PC}}\sigma_{11}^{(PC)}(1 - \nu_{PC}) = \epsilon_0 \quad (9)$$

Grouping equations 8, 9 and 5, we have three equations and three unknowns. ϵ_0 , $\sigma_{11}^{(Al)}$ and $\sigma_{11}^{(PC)}$ can be solved.

$$\epsilon_{11}^{(Al)} = \alpha_{Al}\Delta \quad (10)$$

$$T + \frac{1}{E_{Al}}\sigma_{11}^{(Al)}(1 - \nu_{Al}) = \epsilon_0\epsilon_{11}^{(PC)} = \alpha_{PC}\Delta \quad (11)$$

$$T + \frac{1}{E_{PC}}\sigma_{11}^{(PC)}(1 - \nu_{PC}) = \epsilon_0 2\sigma_{11}^{(Al)}t_{(Al)} + \sigma_{11}^{(PC)}t_{(PC)} = 0 \quad (12)$$

The values of $\epsilon_{33}^{(Al)}$ and $\epsilon_{33}^{(PC)}$ can be found from the constitutive equation

$$\epsilon_{33}^{(Al)} = \alpha_{Al}\Delta T + \frac{1}{E_{Al}}(-\nu_{Al}\sigma_{11}^{(Al)} - \nu_{Al}\sigma_{22}^{(Al)}) \quad (13)$$

$$\epsilon_{33}^{(PC)} = \alpha_{PC}\Delta T + \frac{1}{E_{PC}}(-\nu_{PC}\sigma_{11}^{(PC)} - \nu_{PC}\sigma_{22}^{(PC)}) \quad (14)$$

We know,

$$\begin{aligned} E^{(Al)} &= 72GPa \\ \nu^{(Al)} &= 0.3 \\ \alpha^{(Al)} &= 23 \times 10^{-6}/^\circ K \\ E^{(PC)} &= 2.5GPa \\ \nu^{(Al)} &= 0.4 \\ \alpha^{(Al)} &= 65 \times 10^{-6}/^\circ K \\ \Delta T &= 50^\circ C \end{aligned}$$

From Substitute all these values to the equations above, we can get

$$\begin{aligned} \sigma_{11}^{(Al)} &= 104MPa \\ \sigma_{22}^{(Al)} &= 104MPa \\ \sigma_{11}^{(PC)} &= -12.9MPa \\ \sigma_{22}^{(PC)} &= -12.9MPa \\ \epsilon_0 &= 6.35 \times 10^{-3} \\ \epsilon_{33}^{(Al)} &= 2.83 \times 10^{-4} \\ \epsilon_{33}^{(PC)} &= 7.38 \times 10^{-3} \end{aligned} \quad (15)$$

- The total thickness change includes the thickness change from two Al layers and the PC layer.

$$\begin{aligned}
\Delta t &= 2\Delta t^{(Al)} + \Delta t^{(PC)} \\
&= 2t^{(Al)}\epsilon_{33}^{(Al)} + t^{(PC)}\epsilon_{33}^{(PC)} \\
&= 2 \times 0.25 \times 2.83 \times 10^{-4} + 4 \times 7.38 \times 10^{-3} \\
&= 0.0297mm
\end{aligned}$$

Problem 4 (30 points)

- A loading diagram (fig 2) shows that this is a simple case of a beam in bending and being torqued. There is torque on the shaft, but the load from the pulley also puts it into bending. The goal is to reduce stresses in the shaft due to a bending moment and a torque.

There should be two obvious places where stresses could be high. The first place is at the maximum moment due to bending(point A). The second place is where a stress concentration would most likely occur: the abrupt change in diameter (point B). Thus the goal is now more narrowly defined: reduce bending moment at point A, and reduce the stress concentration at point B.

To reduce the bending moment at A, first realize that you are GIVEN the diameters d and D , the power going into the shaft (*Torque * angularvelocity*), and the pulley itself as design constraints. Look at the diagram to see the only design parameter left as a variable is the DISTANCE FROM THE PULLEY TO THE BEARING (L). The bearing constrains the shaft to have zero displacement at that point and zero slope, hence it is a cantilevered boundary condition. Realizing this is a cantilevered beam, the moment at point A is:

$$M = L(T_1 + T_2) \tag{16}$$

Where $T_1 + T_2$ is simply the force from the belt on the pulley. Regardless the only variable in this case is L, and to reduce the bending stress you should MINIMIZE L .

To reduce stress concentrations, shape transitions should be smooth. Because a torsional loading and bending loading exists, a large radius at point B will help to reduce the stress concentration at the shoulder (see fig 3 for a plot of shear stress distribution with a stress concentration). The stress concentration for a shaft in tension with a shoulder is:

$$K_t \approx 1 + \alpha \sqrt{\frac{d}{\rho}}$$

, where ρ is the radius of the shoulder and d is the smaller radius of the shaft. α is a constant. Please see Dowling, p790 for a plot of the stress concentration for a bending moment. The larger the radius, the better. Of course the effect of large radius and short L are competing against each other (larger R means longer L), and there is probably an optimum radius for a minimum stress, depending on what kind of failure you are worried about.

The point of this problem is the two "driving forces" of this problem are the stress concentration for torsion and bending and the bending moment on the shaft. All other effects, such as friction in the bearing (affecting the torque), position of the second bearing, are secondary. Choosing a shaft with a larger G will affect the angular deflection in the shaft: the strains. In this case we are worried about stresses. The stresses are determined from equilibrium, not G .

- From the compatibility equation

$$2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} \quad (17)$$

we can verify whether or not the polynomial expression is exact.

In the problem,

$$\begin{aligned} \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} &= 2 \frac{x_1^2}{l^4} \\ \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} &= 2 \frac{x_2^2}{l^4} \\ \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} &= \frac{x_1^2 + x_2^2}{l^4} \end{aligned} \quad (18)$$

they indeed satisfy the compatibility relation of eqn 17

- **Limitations of Linear Elastic Analysis.**

- Material is in plastic regime (cannot be predicted by linear elasticity)
- Material fractures (cannot be predicted by linear elasticity)
- Large deformations in material (LINEAR elasticity is valid only for small strains and rotations where the second-order (non linear) terms can be neglected)
- Instability phenomena, e.g. Buckling. Stability of equilibrium requires additional considerations of the nonlinearly deformed configurations.
- Time dependant behavior: viscoelasticity, creep, etc.

The first pillar of Solid Mechanics is Displacements (strains). This is associated with the 3rd limitation above.

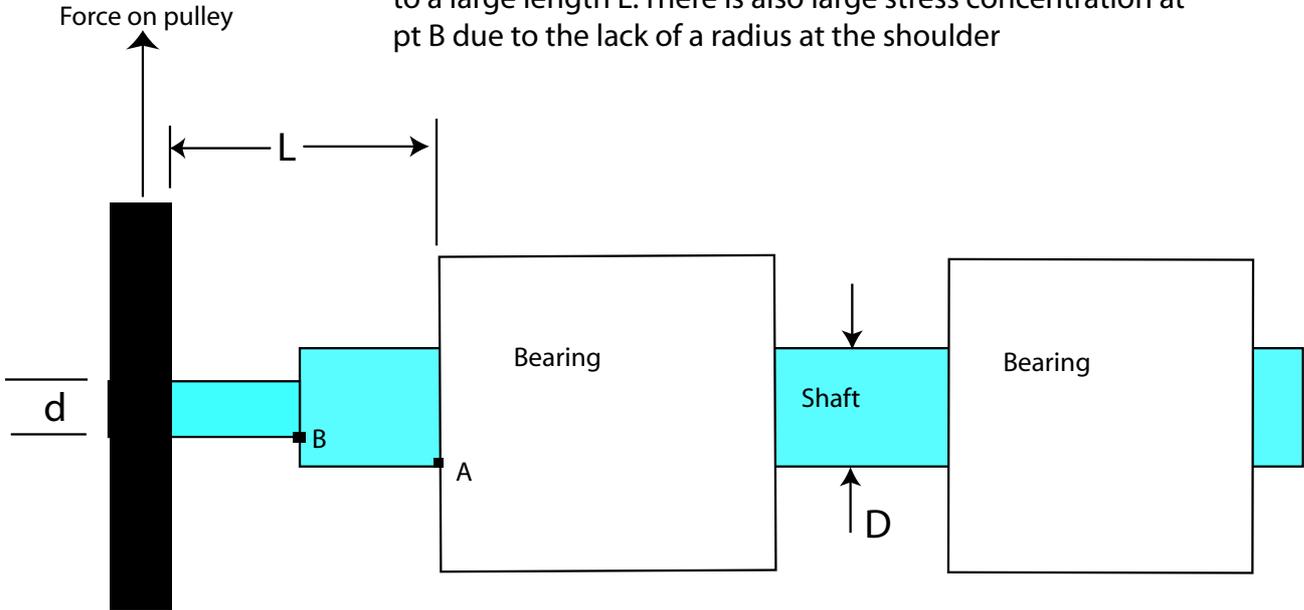
The second pillar of Solid Mechanics is the constitutive relations between displacements and forces (strains and stresses). The relations between stress and strain is not

the same for materials undergoing plasticity when compared to elasticity, as well as materials that creep. The relations between material separation (fracture) and loading is also not covered in linear elasticity.

The third pillar of solid mechanics is forces and equilibrium. Buckling is a phenomenon that results when forces are in equilibrium, but the system is not stable. Linear elasticity acknowledges equilibrium but not stability, at least without consideration of the non-linear geometric terms associated with the bucking mode.

Prob.4a

Bad Design: Has larger moment at points A and B due to a large length L. There is also large stress concentration at pt B due to the lack of a radius at the shoulder



Good Design: By minimizing the distance L, the moments have been reduced and by maximizing the radius at the shoulder (pt b), we have reduced stress concentrations due to an abrupt change in diameter. By reducing the maximum stresses, the likelihood of failure is reduced and thus the design is better.

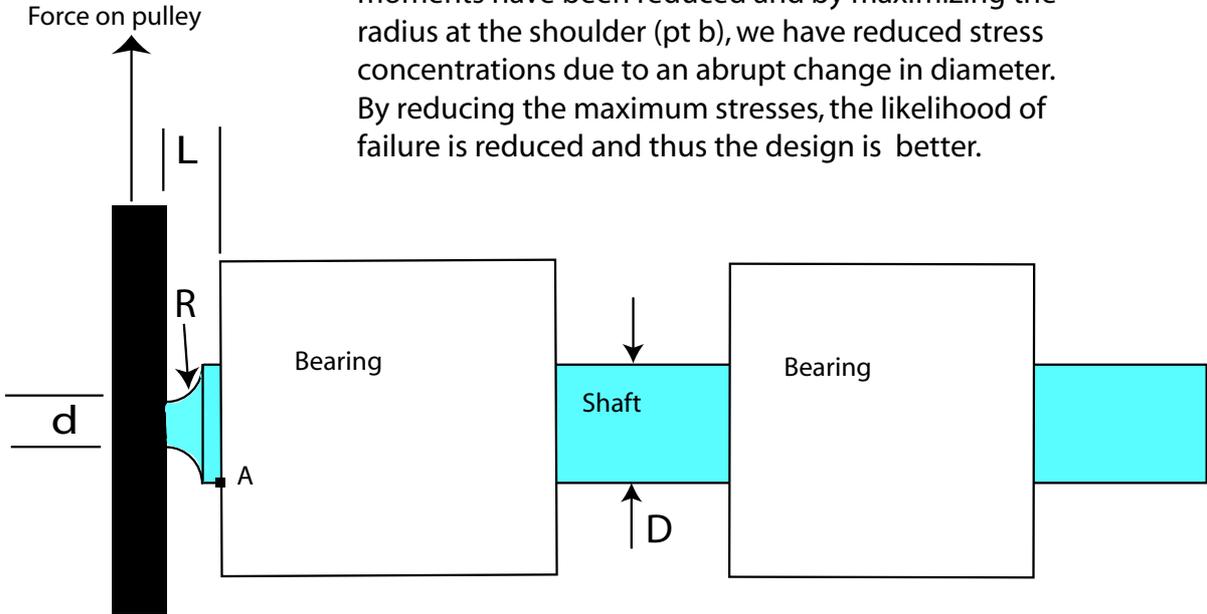


Figure 2: Problem 4a Design layouts

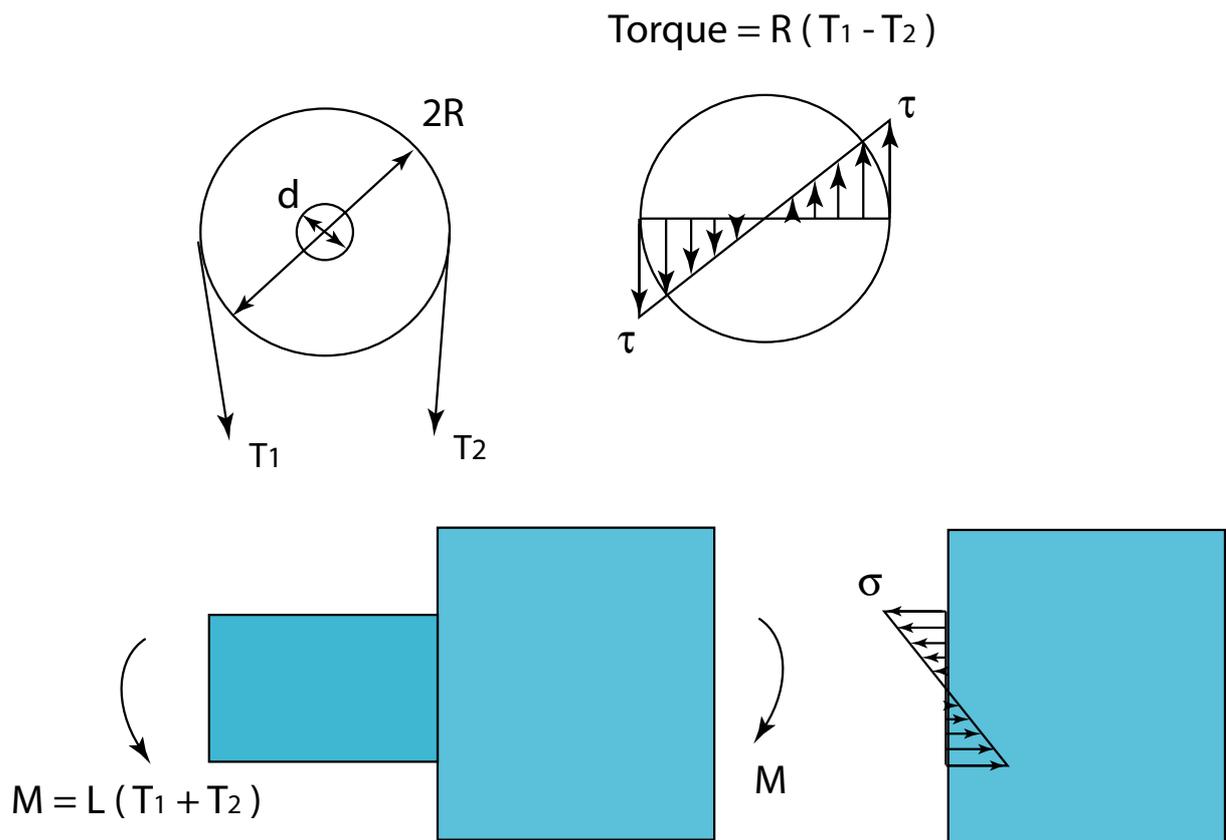


Figure 3: Stress distribution in the shaft