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2.002 MECHANICS AND MATERIALS II
SOLUTIONS FOR HOMEWORK NO. 3

Problem 1 (20 points)
(a) The equilibrium equations are

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 = 0 \tag{1}$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = 0 \tag{2}$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = 0 \tag{3}$$

All shear stresses are zero. Furthermore, the gravitational body force loading b has one non-zero component only, in the direction of e_3. Therefore, from the third equilibrium equation we get:

$$\frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = 0 \Leftrightarrow \frac{\partial \sigma_{33}}{\partial x_3} = -\rho b_3 \tag{4}$$

We also know that $\sigma_{33}(\mathbf{x}) = -p(\mathbf{x})$ and $b_3 = -g$. We can substitute these into Equation 4 and integrate both sides with respect to x_3

$$-\int_0^{x_3} \frac{dp(x)}{dx_3} dx_3 = \int_0^{x_3} -\rho b_3 dx_3 \Leftrightarrow -\int_{p_0}^{p(x)} dp(x) = \rho g x_3 \Leftrightarrow -p(x_3) + p_0 = \rho g x_3 \tag{5}$$

$$p(x_3) = p_0 - \rho g x_3 \tag{6}$$

Note that p is only a function of x_3.

(b) 1. The traction vector on a surface can be found by multiplying the stress with the unit outward normal vector on that surface. In this case, the normal vector is

$$n_l = \begin{Bmatrix} -\cos\theta \\ 0 \\ -\sin\theta \end{Bmatrix} \tag{7}$$

The traction vector is then

$$t_l = \begin{bmatrix} -p(x) & 0 & 0 \\ 0 & -p(x) & 0 \\ 0 & 0 & -p(x) \end{bmatrix} \begin{Bmatrix} -\cos\theta \\ 0 \\ -\sin\theta \end{Bmatrix} \Leftrightarrow t_l = \begin{Bmatrix} p(x)\cos\theta \\ 0 \\ p(x)\sin\theta \end{Bmatrix} \tag{8}$$

2. Action and reaction, forces need to balance at the interface.

$$t_l = \begin{Bmatrix} -p(x)\cos\theta \\ 0 \\ -p(x)\sin\theta \end{Bmatrix} \tag{9}$$

3.

$$t_d = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{Bmatrix} \tag{10}$$

which gives 3 linear equations involving σ_{ij}

$$-p\cos\theta = \sigma_{11}\cos\theta + \sigma_{13}\sin\theta \tag{11}$$

$$0 = \sigma_{21}\cos\theta + \sigma_{23}\sin\theta \tag{12}$$

$$-p\sin\theta = \sigma_{31}\cos\theta + \sigma_{33}\sin\theta \tag{13}$$

Problem 2 (20 points)

$$\epsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\delta_{ij}\sum_{k=1}^{3}\sigma_{kk}\right) + \alpha\Delta T\delta_{ij} \tag{14}$$

The procedure is similar to what was discussed in class for elastic constitutive relations without thermal effects. The idea is to eliminate $\sum_{k=1}^{3}\sigma_{kk}$ from the given equation and solve for σ_{ij}. To do this, take the sum of both sides of the given equation as follows

$$\sum_{i=1}^{3}\epsilon_{ii} = \frac{1+\nu}{E}\left(\sum_{j=1}^{3}\sigma_{jj} - \frac{\nu}{1+\nu}\sum_{k=1}^{3}\sigma_{kk}\sum_{i=1}^{3}\delta_{ii}\right) + \alpha\Delta T\sum_{j=1}^{3}\delta_{jj} \tag{15}$$

The indexes can be anything in this case (i, j, or k) and don't affect the result. $\sum_{j=1}^{3}\delta_{jj}$ is simply the sum of the diagonal elements of the (3x3) identity matrix. Thus, $\sum_{j=1}^{3}\delta_{jj} = 3$. Therefore, the equation becomes

$$\sum_{i=1}^{3}\epsilon_{ii} = \frac{1+\nu}{E}\left(\sum_{j=1}^{3}\sigma_{jj} - 3\frac{\nu}{1+\nu}\sum_{k=1}^{3}\sigma_{kk}\right) + 3\alpha\Delta T \tag{16}$$

Solving for $\sum_{k=1}^{3}\sigma_{kk}$ yields

$$\sum_{k=1}^{3}\sigma_{kk} = \frac{E}{1-2\nu}\left(\sum_{i=1}^{3}\epsilon_{ii} - 3\alpha\Delta T\right) \tag{17}$$

We can now substitute this expression into Equation 14 and solve for σ_{ij} to get

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\sum_{i=1}^{3}\epsilon_{ii} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\delta_{ij}\right) \tag{18}$$

To evaluate the stress components for the given data, use any commercially available software to carry out the matrix operations, such as MATLAB. The stress tensor is

$$\sigma = \begin{bmatrix} -0.6704 & 0.0808 & -0.0323 \\ 0.0808 & -0.2665 & 0 \\ -0.0323 & 0 & -4281 \end{bmatrix} GPa \tag{19}$$

Problem 3 (30 points)

$$\epsilon_{ij} = \epsilon_{ij}^{(\text{thermal})} + \epsilon_{ij}^{(\text{mechanical})} = \alpha\,\Delta T\,\delta_{ij} + \epsilon_{ij}^{(\text{mechanical})} \tag{1}$$

$$\epsilon_{ij} = \alpha\,\Delta T\,\delta_{ij} + \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\left(\sum_{k=1}^{3}\sigma_{kk}\right)\right] \tag{2}$$

This equation can be inverted to give

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\left(\sum_{k=1}^{3}\epsilon_{kk}\right)\delta_{ij} - \frac{(1+\nu)}{(1-2\nu)}\alpha\,\Delta T\,\delta_{ij}\right] \tag{3}$$

1. The strain components are determined by the derivatives of displacement components. Since the displacement should be continuous in the interface separating surface layer from substrate, the in-plane strain components in the thin surface layer should correspond to those in the substrate, too. Therefore, $\epsilon_{11} = \epsilon_{22} = \epsilon_{12} = 0$.

2. The problem stipulates that the thick substrate undergoes negligible total strain and temperature change. It means $\Delta T \doteq 0$ and $\epsilon_{ij(\text{substrate})} \doteq 0_{ij}$.

   The constitutive equation (3) gives the relation between stress components and strain components. In the equation, $\sigma_{ij(substrate)}$ can be calculated by substituting $\epsilon_{ij(\text{substrate})}$ and ΔT(which are all zeros).

$$\sigma_{ij(\ \text{substrate})} = 0_{ij}$$

3. We know that the surface layer has a boundary condition at surface $x_3 = 0$ which remains traction-free ($\mathbf{t^n} = \mathbf{0}$).

$$\mathbf{n} = n_1\mathbf{e_1} + n_2\mathbf{e_2} + n_3\mathbf{e_3} = \mathbf{e_3}$$

$$n_1 = n_2 = 0, \qquad n_3 = 1$$

$$\begin{aligned}
t_1 &= \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 = 0 \Rightarrow \sigma_{13} = 0 &(4)\\
t_2 &= \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 = 0 \Rightarrow \sigma_{23} = 0 &(5)\\
t_3 &= \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 = 0 \Rightarrow \sigma_{33} = 0 &(6)
\end{aligned}$$

The in-plane components ϵ_{11}, ϵ_{22}, and ϵ_{12} are zero.

By the equation (3),

$$\sigma_{12} = \sigma_{21} = \frac{E}{1+\nu}[0+0+0] = 0 \tag{7}$$

Invoking the symmetry of the stress tensor, $\sigma_{ij} = \sigma_{ji}$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

In order to justify the sign of the non-zero stress components, σ_{11} and σ_{22}, we can imagine that, when the temperature increases, the surface layers tends to extend while the substrate tends to prevent it from deforming by shrinking it. Therefore the non-zero stress components would be compression stresses. And similarly, when the temperature decreases, the two in-plane components are tensile stresses. The non-zero components σ_{11} and σ_{22} always have opposite sign from that of the change of temperature(ΔT).

4. By the equation (2) and the in-plane strain continuity($\epsilon_{11} = \epsilon_{22} = \epsilon_{12} = 0$),

$$\begin{aligned}
\epsilon_{11} &= \alpha\,\Delta T\,\delta_{11} + \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\delta_{11}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right]\\
&= \alpha\,\Delta T + \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\left(\sigma_{11}+\sigma_{22}+0\right)\right] = 0
\end{aligned}$$

$$\epsilon_{11} = \alpha\,\Delta T + \frac{1}{E}\left[\sigma_{11} - \nu\sigma_{22}\right] = 0 \tag{9}$$

$$\begin{aligned}
\epsilon_{22} &= \alpha\,\Delta T\,\delta_{22} + \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu\delta_{22}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right]\\
&= \alpha\,\Delta T + \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu\left(\sigma_{11}+\sigma_{22}+0\right)\right] = 0
\end{aligned}$$

$$\epsilon_{22} = \alpha\,\Delta T + \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{11}\right] = 0 \tag{10}$$

On solving equation (9) and (10),

$$\sigma_{11} = \sigma_{22} = -\frac{\alpha\Delta T E}{1-\nu}$$

$$\epsilon_{11} = \epsilon_{12} = \epsilon_{22} = 0, \quad \sigma_{11} = \sigma_{22} = -\frac{\alpha\Delta T E}{1-\nu}$$

Since all the components of the stress tensor in equation (8) are known, the strain tensor can be calculated by the equation (2).

$$\begin{aligned}
\epsilon_{13} &= \alpha\,\Delta T\,\delta_{13} + \frac{1}{E}\left[(1+\nu)\sigma_{13} - \nu\delta_{13}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right] = 0 + \frac{1}{E}\left[0-0\right] = 0 \\
\epsilon_{23} &= \alpha\,\Delta T\,\delta_{23} + \frac{1}{E}\left[(1+\nu)\sigma_{23} - \nu\delta_{23}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right] = 0 + \frac{1}{E}\left[0-0\right] = 0 \\
\epsilon_{33} &= \alpha\,\Delta T\,\delta_{33} + \frac{1}{E}\left[(1+\nu)\sigma_{33} - \nu\delta_{33}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right] \\
&= \alpha\,\Delta T\, + \frac{1}{E}\left[0 + \nu\left(\frac{2\alpha\Delta T E}{1-\nu}\right)\right] = \alpha\Delta T\frac{1+\nu}{1-\nu}
\end{aligned}$$

$$[\epsilon_{ij}] = \alpha\Delta T\frac{1+\nu}{1-\nu}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

5. The Mises equivalent stress σ is given by

$$\overline{\sigma} = \sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2\right]} \tag{12}$$

$$\overline{\sigma} = \frac{E}{1-\nu}\alpha\,|\Delta T| \tag{13}$$

To prevent yielding, it is required that $\overline{\sigma} \leqslant \sigma_y$.

$$|\Delta T| \leqslant \frac{(1-\nu)\sigma_y}{E\alpha}$$

Problem 4 (30 points)
In uniaxial loading, only one stress component is non-zero. Let's assume uniaxial loading in the 1-direction. The stress tensor is then

$$\sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

The stress deviator tensor is then

$$\sigma^{(dev)} = \begin{bmatrix} \frac{2}{3}\sigma_{11} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{11} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{11} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\Sigma & 0 & 0 \\ 0 & -\frac{1}{3}\Sigma & 0 \\ 0 & 0 & -\frac{1}{3}\Sigma \end{bmatrix} \tag{21}$$

The double summation in the Mises stress equation is calculated by squaring each component of the deviatoric stress tensor and adding them all up. More specifically,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}^{(dev}\sigma_{ij}^{(dev} = (\frac{2}{3}\sigma_{11})^2 + (\frac{1}{3}\sigma_{11})^2 + (\frac{1}{3}\sigma_{11})^2 = \frac{2}{3}\Sigma^2 \tag{22}$$

Therefore, the Mises stress becomes

$$\bar{\sigma} = \sqrt{\frac{2}{3}\frac{3}{2}\Sigma^2} = |\Sigma| \tag{23}$$

Hydrostatic pressure only has normal components, ie it doesn't have shear components. As a result, the stress components that are affected by the addition/subtraction of a uniform hydrostatic pressure p are σ_{11}, σ_{22} and σ_{33}, which become in the case of addition, $\sigma_{11}+p$, $\sigma_{22}+p$ and $\sigma_{33}+p$ accordingly. The deviatoric stress components are then

$$\sigma_{11}^{(dev)} = \sigma_{11} + p - \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}+3p) = \sigma_{11} - \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})$$

$$\begin{aligned}
\sigma_{12}^{(dev)} &= \sigma_{12} \\
\sigma_{13}^{(dev)} &= \sigma_{13} \\
\sigma_{21}^{(dev)} &= \sigma_{21} \\
\sigma_{22}^{(dev)} &= \sigma_2 + p - \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}+3p) = \sigma_{22} - \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\
\sigma_{23}^{(dev)} &= \sigma_{23} \\
\sigma_{31}^{(dev)} &= \sigma_{31} \\
\sigma_{32}^{(dev)} &= \sigma_{32} \\
\sigma_{33}^{(dev)} &= \sigma_3 + p - \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}+3p) = \sigma_3 - \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})(24)
\end{aligned}$$

which is equal to the original stress deviator tensor.

In plane stress

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

Thus, the stress deviator tensor is

$$\sigma^{(dev)} = \begin{bmatrix} \frac{2}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} & \sigma_{12} & 0 \\ \sigma_{12} & \frac{2}{3}\sigma_{22} - \frac{1}{3}\sigma_{11} & 0 \\ 0 & 0 & -\frac{1}{3}(\sigma_{11} + \sigma_{22}) \end{bmatrix} \tag{26}$$

Then, the Mises stress becomes

$$\bar{\sigma} = \sqrt{\frac{3}{2}3\sigma_{12}^2 + \frac{3}{2}\frac{1}{9}(4\sigma_{11}^2 + \sigma_{22}^2 - 4\sigma_{11}\sigma_{22}) + \frac{3}{2}\frac{1}{9}(4\sigma_{22}^2 + \sigma_{11}^2 - 4\sigma_{11}\sigma_{22}) + \frac{3}{2}\frac{1}{9}(\sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{11}\sigma_{22})} \tag{27}$$

After some manipulation, we get

$$\bar{\sigma}^2 = 3\sigma_{12}^2 + \sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} \tag{28}$$

Finally, in the case of pure shear, the stress tensor is

$$\sigma = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

The stress deviator tensor is identical to the stress tensor since pure shear does not cause volumetric changes (only shape changes). Thus, $\sigma = \sigma^{dev}$. The Mises stress is

$$\bar{\sigma} = \sqrt{\frac{3}{2}2\tau^2} \Leftrightarrow |\tau| = \frac{\bar{\sigma}}{\sqrt{3}} \tag{30}$$

As mentioned in the problem statement, yielding occurs when $\bar{\sigma} = \sigma_y$. Therefore, $\tau| = \frac{\sigma_y}{\sqrt{3}}$. Furthermore, the only non-zero variable in the Tresca condition is $\sigma_{12} = \tau$. Thus, the first inequality yields $|\tau| \leq \frac{\sigma_y}{2}$ and the second one $|\tau| \leq \sigma_y$. Thus, the Tresca condition predicts yielding when $|\tau| = \frac{\sigma_y}{2}$.