MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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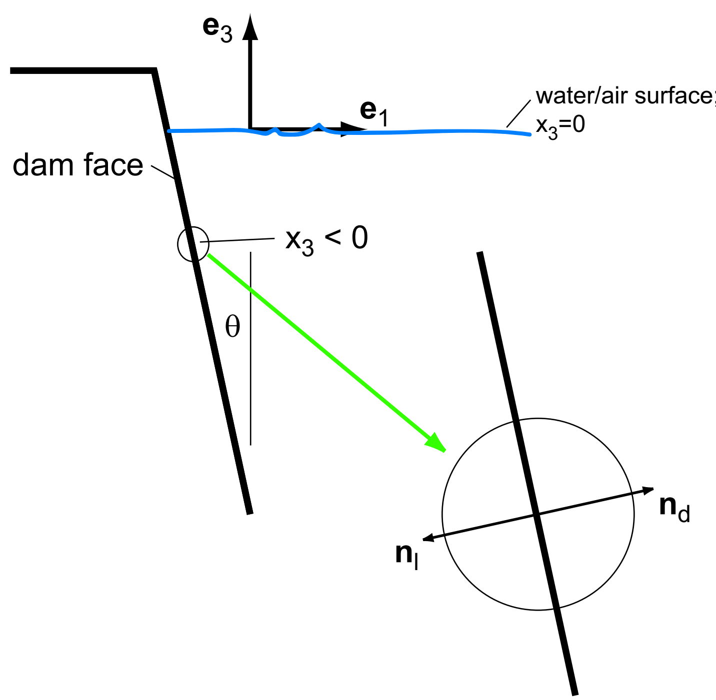
2.002 MECHANICS AND MATERIALS II
HOMEWORK NO. 4

Distributed: Wednesday, March 3, 2004
Due: Wednesday, March 10, 2004

Problem 1 20 Points



(a) The state of stress at position $\mathbf{x}$ in a fluid at rest (say, a liquid) can be characterized by

$$\sigma_{ij}(\mathbf{x}) = -p(\mathbf{x})\,\delta_{ij},$$

where p is the [fluid] pressure at $\mathbf{x}$. Assume that the liquid has a constant mass density, ρ, and assume further that the fluid is subject to a gravitationally-induced body force loading (per unit mass) of magnitude

$$\mathbf{b} = -g\,\mathbf{e}_3,$$

where the cartesian basis vector $\mathbf{e}_3$ points "up." Let atmospheric [air] pressure at the surface of the liquid be p_0 (elevation of air/liquid surface: $x_3 = 0$) . Using the appropriate equilibrium equations, show that the pressure at generic elevation $x_3 < 0$ is given by

$$p(x_1,\, x_2,\, x_3) = p_0 - \rho\, g\, x_3.$$

(b) A long, straight dam holds the water in place. The dam extends along the $\mathbf{e}_2$ direction, and the planar surface of the dam makes an angle of θ with respect to the vertical, as shown. At a point on the liquid/dam interface that is at elevation $x_3 < 0$:

1. evaluate the traction vector exerted **by** the dam **on** the fluid surface. The outward normal vector on the liquid is $\mathbf{n}_l$, as shown.

2. explain why the traction vector acting on the surface of the dam (that is, the traction exerted **by** the fluid **on** the surface element of the dam) is equal and opposite to the previously-determined traction vector in part (1). The outward normal to the surface of the dam is $\mathbf{n}_d$, as shown.

3. use the traction vector from part (2) to express 3 linear equations involving the cartesian stress components σ_{ij} in the dam at elevation x_3.

Problem 2 (20 Points)

The isotropic linear thermal/elastic constitutive relations can be expressed in the compact notation

$$\epsilon_{ij} = \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\left(\sum_{k=1}^{3}\sigma_{kk}\right)\right] + \alpha\Delta T\,\delta_{ij}.$$

Here the total strain components (ϵ_{ij}) are expressed in terms of the stress components (σ_{ij}) and the temperature change (ΔT).

Derive a consistent expression for the stress components, σ_{ij}, in terms of the strain components and the temperature change. Evaluate those stress components when the strain tensor is

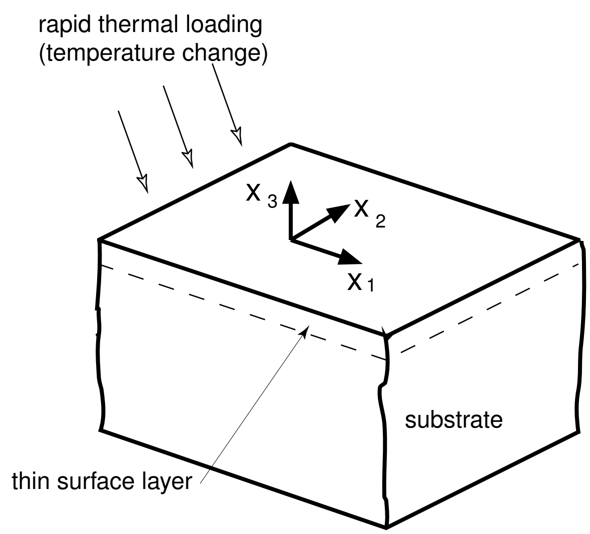
$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} -0.001 & 0.0005 & -0.0002 \\ 0.0005 & 0.0015 & 0.0 \\ -0.0002 & 0.0 & 0.0005 \end{bmatrix},$$

and the temperature change is $\Delta T = 100\,C$. Material properties include $E = 210\,GPa$, $\nu = 0.3$, and $\alpha = 12 \times 10^{-6}/\,C$

Problem 3 (30 Points))

The surface $x_3 = 0$ of a large, flat body made of a material having elastic Young's modulus E, Poisson ratio ν, and coefficient of thermal expansion α is subjected to a rapid thermal shock. [1] The results in either case can be summed up by the following observations:

- In a thin layer on and immediately below the surface $x_3 = 0$:
  - the temperature rapidly changes by an amount "ΔT", where $\Delta T \equiv T_{\text{after}} - T_{\text{before}}$;
  - the surface $x_3 = 0$ remains traction-free.
- In the thick substrate beneath the surface layer:
  - the temperature changes negligibly: $\Delta T \doteq 0$;
  - the total strain remains very small: $\epsilon_{ij} \doteq 0_{ij}$.



[1]This can be accomplished, *e.g.*, by rapidly bringing the surface into contact with a flame or laser irradiation (heating); alternatively, a very cold liquid may rapidly flow over the surface (cooling).

Assuming that the response of the material during the shock is isotropic linear thermal/elastic:

1. Argue why the in-plane strain components ϵ_{11}, ϵ_{22}, and ϵ_{12} in the thin surface layer should match the corresponding values in the substrate; namely, $\epsilon_{11} = \epsilon_{22} = \epsilon_{12} = 0$. (Hint: what would happen to the displacement field components u_i if any of these strain components *were* discontinuous across the interface separating surface layer from substrate?)

2. Evaluate all components of the stress tensor σ_{ij} in the substrate.

3. Evaluate all components of the stress tensor σ_{ij} in the thin surface layer. Give "physical" reasons to justify the sign of any non-zero stress components, relative to the sign of ΔT.

4. Evaluate all components of the strain tensor ϵ_{ij} in the surface layer. Give "physical" reasons to justify the sign of any non-zero strain components, relative to the sign of ΔT.

5. Suppose that yielding of the material begins whenever the Mises stress measure $\overline{\sigma}$ reaches the value σ_y, which is the tensile yield strength of the material. Calculate the maximum value of $|\Delta T|$ which can be applied to the surface layer without having the Mises stress exceed the value "σ_y".

Recall that in an isotropic linear thermal/elastic material, the relation between temperature change, ΔT, stress components, σ_{ij}, and strain components, ϵ_{ij}, is

$$\begin{aligned} \epsilon_{ij} &= \epsilon_{ij}^{(\text{thermal})} + \epsilon_{ij}^{(\text{mechanical})} \\ &= \alpha\,\Delta T\,\delta_{ij} + \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\left(\sum_{k=1}^{3}\sigma_{kk}\right)\right] \end{aligned}$$

You may consult your [correct] answer to Problem 2, above, in order to find an appropriate "inverted" form giving the stress components σ_{ij} in terms of the strain components ϵ_{ij} and the temperature change, ΔT.

Note: In this and in many other problems, you have information about some (but not all!) of the stress components, and about some (but not all!) of the strain components. In this case, you must engage in "hand-to-hand combat" with the set of linear equations comprising the stress/strain/temperature relations, obtaining new expressions for those components that you do not know, *a priori*, in terms of those components that you do know .....

Problem 4 (30 Points)

The **Mises equivalent tensile stress** is a non-negative scalar measure of a multi-axial stress state, σ_{ij}, that is used to asses the proximity to yielding in ductile metals and polymers at sufficiently low temperatures. Let the uniaxial tensile yield strength of a material be given by σ_y; then, if that material is under a general multi-axial stress state σ_{ij} whose Mises tensile equivalent stress, $\bar{\sigma}$, satisfies

$$\bar{\sigma} < \sigma_y,$$

no plastic deformation is forthcoming: the material will respond elastically. Should $\bar{\sigma} = \sigma_y$, plastic deformation under the multiaxial stress state is imminent.

Thus, it is important to be able to calculate the Mises tensile equivalent stress. The most fundamental definition of $\bar{\sigma}$ first involves calculation of the **stress deviator tensor** corresponding to σ_{ij}. Components of the stress deviator are defined by

$$\sigma_{ij}^{(\text{dev})} \equiv \sigma_{ij} - \frac{1}{3}\,\delta_{ij}\,\left(\sum_{k=1}^{3}\sigma_{kk}\right).$$

The Mises equivalent tensile stress is given by

$$\bar{\sigma} \equiv \sqrt{\frac{3}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}^{(\text{dev})}\,\sigma_{ij}^{(\text{dev})}}.$$

Clearly, it is a "root mean square" measure of the stress deviator tensor components (square root of the sum of the squares of the components).

- Show that, for uniaxial tensile/compressive stress of magnitude "Σ" along any coordinate axis, $\bar{\sigma} = |\Sigma|$.
- Show that addition or subtraction of a uniform hydrostatic pressure to a given stress state, σ_{ij}, does not affect the value of $\bar{\sigma}$.
- Show that, in plane stress, where the only non-zero components of the stress tensor are σ_{11}, σ_{22}, and $\sigma_{12} = \sigma_{21}$, the Mises tensile equivalent stress satisfies

$$\bar{\sigma}^2 = 3\sigma_{12}^2 + \sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22}.$$

- In 2.001, you may have considered an alternative criterion for the onset of plastic flow under multiaxial stress states, termed the maximum shear stres, or Tresca yield criterion. In the special case of plane stress given above, the Tresca yield condition comprises the following sets of constraints:

$$2\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \leq \sigma_y.$$

$$\left| \frac{\sigma_{11}+\sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \right| \leq \sigma_y.$$

Show that, in the case of pure shear, $\sigma_{ij} = 0$ except $\sigma_{12} = \sigma_{21} = \tau \neq 0$, the Mises yield criterion predicts that yield will occur when $|\tau| = \sigma_y/\sqrt{3}$, while the Tresca yield criterion predicts yielding when $|\tau| = \sigma_y/2$. Thie difference is only $\sim 15\%$, and is the largest possible difference in the predictions of yielding between the Tresca and Mise criteria. In general states of stress, computation of the Mises equivalent tensile stress measure is straightforward, but the generalization of the Tresca yield criterion to general states of strees requires more calculations. Also, careful experimentation under multiaxial stress (e.g., using thin-walled tubes under combinations of tension/compression, torsion, and internal pressure) gives results generally closer to the Mises criterion.