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2.002 MECHANICS AND MATERIALS II
SOLUTIONS FOR HOMEWORK NO. 2

Problem 1 (15 points)
The axial stiffness of the structure is:

$$k_{axial} = \frac{F_x}{\delta_x} = \frac{\sigma_{xx} A}{\epsilon_{xx} L} = E\frac{A}{L} \tag{1}$$

where A is the area of the cross section and L is the length of the beam. The bending stiffness is defined as:

$$k_{bending} = \frac{F_y}{\delta_y}, \tag{2}$$

where F_y is the concentrated tip load, and δ_y is the related tip deflection. According to the beam theory, we know that $\delta_y = \frac{F_y L^3}{3EI}$. Thus, the bending stiffness is:

$$k_{bending} = \frac{F_y}{\delta_y} = \frac{F_y}{\frac{F_y L^3}{3EI}} = \frac{3EI}{L^3}. \tag{3}$$

Combining Eq.1 and Eq.3, the ratio of a slender cantilever's bending stiffness to its axial stiffness is:

$$\frac{k_{bending}}{k_{axial}} = \frac{\frac{3EI}{L^3}}{E\frac{A}{L}} = \frac{3I/A}{L^2} = 3(\frac{l_2}{L})^2 \tag{4}$$

where $l_2 \equiv \sqrt{I/A}$. Since $L \gg l_2$, we find the above ratio is extremely small.

For a solid circular cross-section beam with diameter d, we have:

$$I = \frac{\pi d^4}{64} \tag{5}$$

and

$$A = \frac{\pi d^2}{4} \tag{6}$$

Substitution of the A and I into Eq. 4, we have

$$\frac{k_{bending}}{k_{axial}} = \frac{3I/A}{L^2} = \frac{3}{16}(\frac{d}{L})^2 \tag{7}$$

Problem 2 (45 points)
Part A:
No forces and moments are applied to the beam. And at any x, $0 < x < l$, we have:

$$N = F_x = \int \sigma_{xx} dA = 0, \tag{8}$$

and

$$M = \int \sigma_{xx}(-y)dA = 0 \tag{9}$$

The stress is a function of y, $\sigma_{xx} = \sigma_{xx}(y)$, and does not change along the x-direction. The deformation is composed of two parts: the thermal part and the mechanical part, thus the total stain at each point can be expressed as:

$$\epsilon_{(total)} = \epsilon_{(mechanical)} + \epsilon_{(thermal)} \tag{10}$$

where

$$\epsilon_{(mechanical)} = \sigma(y)/E \tag{11}$$

and

$$\epsilon_{(thermal)} = \alpha \Delta T = \alpha(\Delta T_0 - \frac{\Delta T^* y}{h/2}) \tag{12}$$

Meanwhile, since the beam deflects laterally with a constant curvature, $\kappa_{(thermal)}$, the total strain of the beam can also be expressed as:

$$\epsilon_{total} = \epsilon_0 - \kappa_{(thermal)} y \tag{13}$$

where ϵ_0 is the total strain for the mid-surface axial.

$$\epsilon_0 = \frac{\partial u_0(x)}{\partial x} = \epsilon_{0(thermal)} \rightarrow \epsilon_{total} = \epsilon_{0(thermal)} - \kappa_{(thermal)} y \tag{14}$$

From Eq.s 10, 11, 12 and Eq. 14, we have:

$$\epsilon_{0(thermal)} - \kappa_{(thermal)} y = \epsilon_{(mechanical)} + \epsilon_{(thermal)} = \sigma(y)/E + \alpha(\Delta T_0 - \frac{\Delta T^* y}{h/2}) \tag{15}$$

Thus, the stress can be expressed as:

$$\sigma_{xx}(y) = E[\epsilon_{0(thermal)} - \kappa_{(thermal)} y - \alpha(\Delta T_0 - \frac{\Delta T^* y}{h/2})] \tag{16}$$

Substituting Eq. 16 into Eq. 8, we have:

$$Eb \int_{-h/2}^{h/2} [\epsilon_{0(thermal)} - \kappa_{(thermal)} y - \alpha(\Delta T_0 - \frac{\Delta T^* y}{h/2})] dy = 0 \tag{17}$$

The integration of the linear terms in the above equation, $-\kappa_{(thermal)} y$ and $\frac{\Delta T^* y}{h/2}$, will be zero, since the integral range is symmetric (from $-h/2$ to $h/2$). Thus we have:

$$Eb(\epsilon_{0(thermal)} - \alpha \Delta T_0) \int_{-h/2}^{h/2} dy = 0 \rightarrow \epsilon_{0(thermal)} = \alpha \Delta T_0 \tag{18}$$

Another easier way to get this relation is that:

$$\epsilon_{0(thermal)} = \alpha \Delta T(y = 0) \rightarrow \epsilon_{0(thermal)} = \alpha \Delta T_0 \tag{19}$$

Substituting Eq. 16 into Eq. 9 we have:

$$Eb\int_{-h/2}^{h/2}[\epsilon_{0(thermal)} - \kappa_{(thermal)}y - \alpha(\Delta T_0 - \frac{\Delta T^* y}{h/2})](-y)dy = 0 \tag{20}$$

Considering that $\epsilon_{0(thermal)} = \alpha \Delta T_0$, and $\int y^2 dA \equiv I \neq 0$, we get:

$$\kappa_{(thermal)} = \frac{2\Delta T^* \alpha}{h} \tag{21}$$

Part B:

Substitution of Eq. 18 and Eq. 21 into the expression of the stress, Eq. 16, we get:

$$\sigma_{xx}(y) = E[\alpha \Delta T_0 - \frac{2\Delta T^* \alpha}{h}y - \alpha(\Delta T_0 - \frac{\Delta T^* y}{h/2})] = 0 \tag{22}$$

The axial stress in the thermally-loaded cantilever is zero everywhere.

Part C:

The tip deflection caused by the thermal load is:

$$v_{\Delta T}(x = L) = \frac{1}{2}\kappa_{(thermal)}L^2 = \frac{\Delta T^* \alpha}{h}L^2 \tag{23}$$

The tip deflection caused by the reaction force R_{tip} is:

$$v_{R_{tip}}(x = L) = \frac{R_{tip}L^3}{3EI} \tag{24}$$

The actual tip deflection is zero. By applying superposition, we get:

$$0 = v_{\Delta T}(x = L) + v_{R_{tip}}(x = L) \rightarrow \frac{\Delta T^* \alpha}{h}L^2 = -\frac{R_{tip}L^3}{3EI} \rightarrow R_{tip} = -\frac{3EI\alpha\Delta T^*}{hL} \tag{25}$$

Part D:

As shown in Part B, the thermal load does not generate stress, thus the stress field of this tip- restrained thermally-loaded cantilever beam is the same as the beam is only subjected to a tip load of R_{tip}. Since the sign of the tip-load is negative, R_{tip} is downwards and the upper part of the beam is in tension and the lower pare is in compression. The bending moment is expressed as:

$$M(x) = R_{tip}(L - x) \tag{26}$$

The maximum tensile stress is at $(x = 0, y = h/2)$, and value is expressed as:

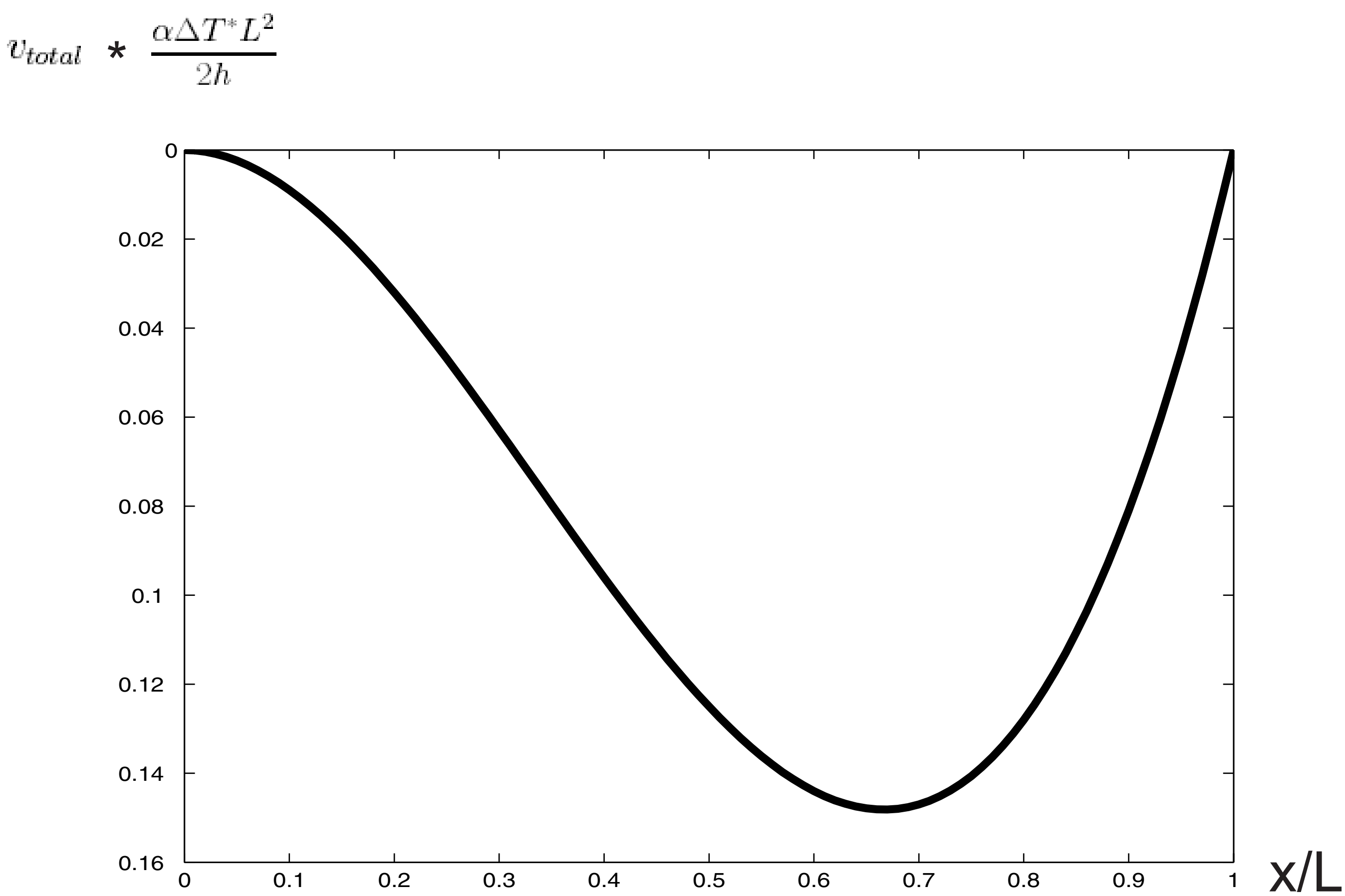
$$\sigma_{max} = -\frac{M\frac{h}{2}}{I} = -\frac{R_{tip}L\frac{h}{2}}{I} = \frac{3}{2}E\alpha\Delta T^* \tag{27}$$

Part E:

The problem requests that $\sigma_{max} < \sigma_y = 350MPa$, substitution of Eq. 25 into the above relation, we have:

$$\sigma_{max} = \frac{3}{2}E\alpha\Delta T^* < \sigma_y \rightarrow \Delta T^* < \frac{2\sigma_y}{3E\alpha} \tag{28}$$

With $E = 210GPa$ and $\alpha = 12 \times 10^{-6}/^{\circ}K$, we find the largest bottom/top difference in temperature change, $\Delta T^* = 92.6^{\circ}K$. So the largest bottom/top difference in temperature change is $2 * \Delta T^* = 185.2^{\circ}K$



$$v_{total} = \frac{1}{2}\kappa_{(thermal)}x^2 + \frac{R_{tip}x^2(3L-x)}{6EI} = \frac{\alpha \Delta T^* L^2}{2h}(\frac{x}{L})^2(\frac{x}{L}-1)$$

Fig 01: The total deflection

Problem 3 (20 points)
The lattice-based prediction of the density of iron is:

$$\rho_{Fe} = \frac{2 \times \frac{55.847}{6.02} \times 10^{-23} g}{0.2866^3 \times 10^{-27} m^3} = 7.88 Mg/m^3 \tag{29}$$

For Sodium chloride, there is one half atom ($4 \times 1/8$) of both sodium (Na) and chlorine (Cl) in each cell. The atomic mass for sodium is $22.99 g/mol$ and $35.453 g/mol$ for chlorine.

$$\rho_{NaCl} = \frac{0.5 \times \frac{(22.99+35.453)}{6.02} \times 10^{-23} g}{r_0^3 m^3} = 2.17 Mg/m^3 \tag{30}$$

Solving the above equation, we got the estimated lattice spacing $r_0 = 0.2818 nm$.

Problem 4 (20 points)

$$S_0 \doteq E r_0 \tag{31}$$

Metal	E (GPa)	a_0 ($\mathring{A}$)	S_0 (N/m)
Ni	214	3.1517	67.446
Al	70	4.0496	28.347
Pt	172	3.9231	67.477
Pd	124	3.8902	48.238
Cu	124	3.6151	44.827
Au	82	4.0786	33.445
Ag	76	4.0862	31.055

We can see that Au and Ag have the similar value in S_0 and Ni and Pt have the similar value in S_0. Conclusion: elements in the same column of the periodic table of elements tend to have similar atomic bond stiffness (S_0)

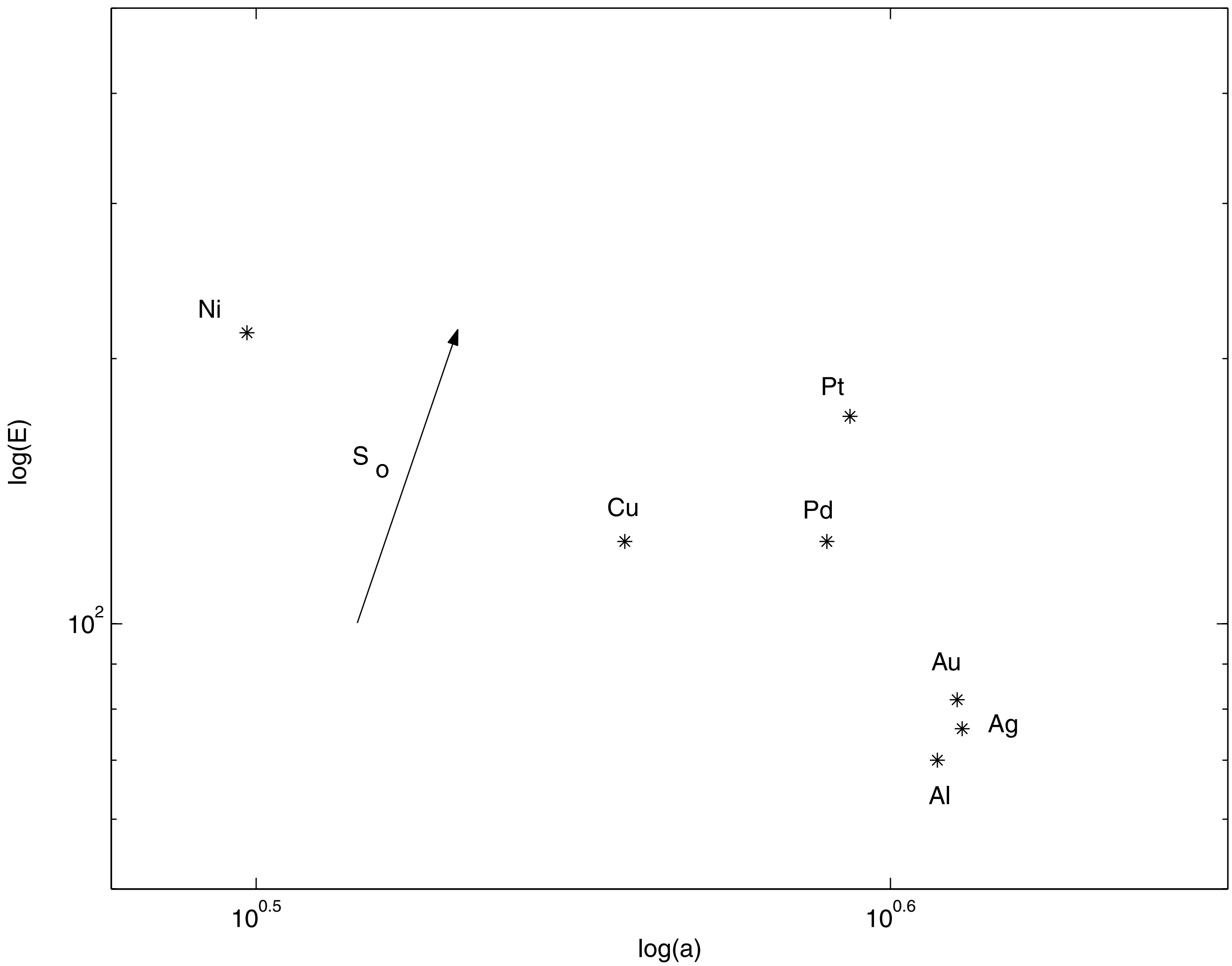


Fig 02: E vs. a_o