MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139

2.002 MECHANICS AND MATERIALS II
SOLUTION for HOMEWORK NO. 1

Distributed: Wednesday, September 10, 2003
Due: Wednesday, September 17, 2003

Problem 1

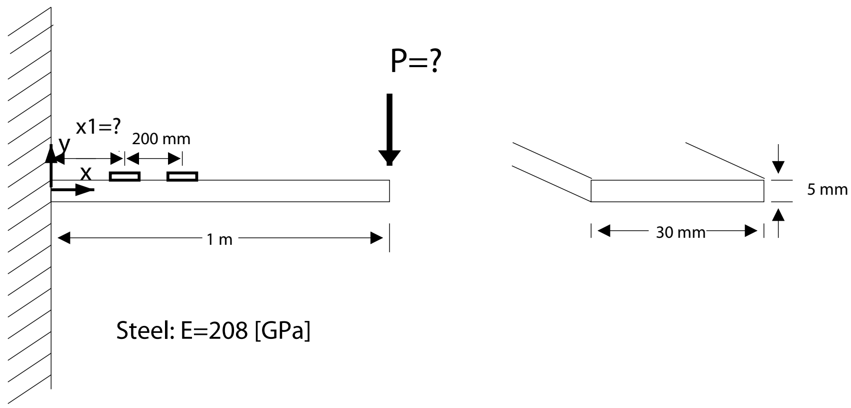


Figure 1: schematic drawing of Problem 1

This beam is under bending and shear. Strain gauges are located at the top surface of the beam. On the top surface, there is zero transverse shear. Thus the beam is only in a stress state due to bending at the location of the strain gauges. We must know how to compute the stress due to a bending moment and relate that stress to the strain in order to solve this problem.

First, the axial stress due to bending is

$$\sigma(x,y) = \frac{-M(x)y}{I} \tag{1}$$

where I and y on the top of surface of the beam are the following

$$I = \frac{bh^3}{12} \tag{2}$$

$$y = \frac{h}{2} \tag{3}$$

M can be found via taking a section cut in the beam and balancing the moments. Be sure to use proper sign convention

$$M(x) = -P(L-x) \tag{4}$$

Re-write (1) using (2), (3), (4)

$$\sigma(x) = \frac{-(-P(L-x))h}{2\frac{bh^3}{12}} = \frac{6P(L-x)}{bh^2} \tag{5}$$

This is in the Linear-Elastic regime so,

$$\sigma(x) = E\epsilon(x) \tag{6}$$

Re-write (5) with (6) to get

$$\epsilon(x) = \frac{6P(L-x)}{Ebh^2} \tag{7}$$

We have two values of strains ϵ_1 and ϵ_2 at positions x_1 and x_2 so we now have two equations (7), (8), with three unknowns x_1, x_2, and P

$$\epsilon_1 = \frac{6P(L-x_1)}{Ebh^2} \tag{8}$$

$$\epsilon_2 = \frac{6P(L-x_2)}{Ebh^2} \tag{9}$$

The third equation needed comes from the geometry condition given in the problem statement

$$x_2 - x_1 = 200 \times 10^{-3} = d \tag{10}$$

Now we just have 3 unknowns and 3 equations so it can be solved anyway you like. One was is shown below

To solve for P, combine equations (8), (9), (10)

$$\epsilon_1 - \epsilon_2 = \frac{6P}{Ebh^2}(x_2 - x_1) = \frac{6Pd}{Ebh^2} \tag{11}$$

Now solve (11) for P, and plug in values. We know

$$\begin{aligned} \epsilon_1 &= 1200 \times 10^{-6} \\ \epsilon_2 &= 900 \times 10^{-6} \end{aligned}$$

then,

$$\begin{aligned} P &= \frac{(\epsilon_1 - \epsilon_2) E b h^2}{6d} \\ P &= \frac{300 \times 10^{-6} \cdot 30 \times 10^{-3}[m] \cdot (5 \times 10^{-3})^2 [m^2] \cdot 208 \times 10^9 [N/m^2]}{6 \cdot 200 \times 10^{-3}[m]} = 39[N] \quad (12) \end{aligned}$$

Knowing P we can get the positions of the strain gauges. Solve (8) for x_1.

$$\begin{aligned} x_1 &= L - \frac{E b h^2 \epsilon_1}{6P} \\ &= 1[m] - \frac{208 \times 10^9 [N/m^2] \cdot 30 \times 10^{-3}[m] \cdot (5 \times 10^{-3})^2 [m^2] \cdot 1200 \times 10^{-6}}{6 \cdot 39[N]} \\ &= 1 - 0.8 = 0.2[m] \quad (13) \end{aligned}$$

Now get x_2

$$x_2 = x_1 + d \quad (14)$$

$$x_2 = 0.2\ m + 0.2\ m = 0.4\ m \quad (15)$$

Problem 2

The natural frequency of a simple harmonic oscillator depends on both the stiffness of the restoring (elastic) member in the system and the mass which is being accelerated/decelerated. For a rigid mass m connected to a massless spring of linear stiffness k (dimensions: force/length), having one end grounded while the other is attached to the moving mass, the natural frequency is simply

$$\omega = \sqrt{\frac{k}{m}} \quad (16)$$

For the first-mode natural frequency of continuous uniform beam, ω_0, we have

$$\omega = \sqrt{\frac{k}{\alpha m}} \quad (17)$$

where $\alpha = 0.24$. To see where it comes from, please refer to section 8 of Lab No.1 handout.

1). Estimate the first-mode natural frequency, ω_0

We are assuming that the antibody coating itself does not appreciably affect the natural frequency of the cantilever. So, based on the geometry of the beam, we have the beam stiffness,

$$k = \frac{P}{\delta} = \frac{3EI}{L^3} = \frac{3 \cdot 100 \times 10^9 [N/m^2] \cdot 4.10 \times 10^{-26} [m^4]}{(100 \times 10^{-6})^3} [m^3] = 1.23 \times 10^{-2} [N/m] \qquad (18)$$

where I is the area moment of inertia of the cross-section, which, for rectangular cross-sections of this orientation, is equal to:

$$I = \frac{bh^3}{12} = \frac{15 \times 10^{-6} [m] \cdot (320 \times 10^{-9})^3 [m^3]}{12} = 4.10 \times 10^{-26} [m^4] \qquad (19)$$

with b the width and h the thickness of the beam.

The mass of this cantilever is equal to its volume times its density, which is,

$$m = \rho V = \rho \cdot L \cdot b \cdot h = 3.1 \times 10^3 [kg/m^3] \cdot 100 \times 10^{-6} [m] \cdot 15 \times 10^{-6} [m] \cdot 320 \times 10^{-9} [m] = 1.49 \times 10^{-12} [kg] \qquad (20)$$

Substitute eqn 18 and eqn 19 into eqn 17, we have the first mode natural frequency of this microfabricated cantilever,

$$\omega = \sqrt{\frac{k}{\alpha m}} = \sqrt{\frac{1.23 \times 10^{-2} [kgm/s^2 \Rightarrow N/m]}{0.24 \cdot 1.49 \times 10^{-12} [kg]}} = 1.85 \times 10^5 [rad/s] \qquad (21)$$

2). Frequency change from the added mass, $\Delta\omega$

According to eqn 18 and eqn 21, the natural frequency of a cantilever is

$$\omega = \sqrt{\frac{3EI}{\alpha L^3 m}} \propto \frac{1}{L^2} \sqrt{\frac{EIL}{m}} \propto \frac{\sqrt{EIL}}{L^2} \frac{1}{\sqrt{m}} \qquad (22)$$

The sought-for bacteria will preferentially attach themselves to the antibody coating on the surface of the cantilever, in the process increasing the vibrating mass of the cantilever by an amount $\Delta m = n_b m_b$, where n_b is the number of bacterium cells that attach, and m_b is the mass of the bacterium cell.

We are assuming the added mass Δm is uniformly distributed along the length of the beam, and further, assume that the presence of the adhered bacteria does not affect the stiffness of the beam. From Eqn 22, the change in frequency resulting from the add mass can be expressed by

$$\omega_0 + \Delta\omega \doteq \frac{\sqrt{EIL}}{L^2}\frac{1}{\sqrt{m+\Delta m}} \doteq \frac{\sqrt{EIL}}{L^2\sqrt{m}}\frac{1}{\sqrt{1+\frac{\Delta m}{m}}} = \frac{\omega_0}{\sqrt{1+\frac{\Delta m}{m}}} \tag{23}$$

For the function of $(1+\frac{\Delta m}{m})^{-\frac{1}{2}}$, if the change of in total mass (Δm) is very small in comparison to the initial beam mass (m), then it can be expanded by taking a Taylor series expansion. The general Taylor series expansion has the form of

$$\begin{aligned} f(x+\Delta x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}\Delta x^n \\ &= f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \frac{f'''(x)}{3!}(\Delta x)^3 + ... \\ &= f(x) + \frac{f'(x)}{1!}\Delta x + O(x^2) \\ &\doteq f(x) + \frac{f'(x)}{1!}\Delta x \end{aligned} \tag{24}$$

In our case,

$$(1+\frac{\Delta m}{m})^{-\frac{1}{2}} \doteq 1 - \frac{1}{2}\frac{\Delta m}{m} \quad \text{if } \frac{\Delta m}{m} \ll 1 \tag{25}$$

Then, the change in frequency resulting from the adding mass can be expressed as the following

$$\omega_0 + \Delta\omega \doteq \omega_0(1 - \frac{1}{2}\frac{\Delta m}{m}) \tag{26}$$

3). Evaluate the change in natural frequency for the bacterium-coated cantilever

For each bacterium cell, we assume it is spherical, with the diameter, $D_b = 1\mu m$. Its density, ρ_b, is equal to that of water, i.e. $1.0 \times 10^3 kg/m^3$. An estimate of each cell's mass is

$$m_b = \rho_b V = \rho_b \frac{4}{3}\pi(\frac{D}{2})^3 \doteq 1.0 \times 10^3[kg/m^3] \cdot \frac{4}{3}\pi(\frac{1\times 10^{-6}}{2})^3[m^3] = 5.24 \times 10^{-16}[kg] \tag{27}$$

The total mass of all the bacterium cells, which is the "added mass", is

Image removed due to copyright considerations.

Figure 2: Scanning electron microscope images of *E. coli* bacteria attached to various microfabricated resonating cantilever beams. (from: Ilic, et al., *Applied Physics Letters*, **77**, #3, 2000, 450-452.

$$\Delta m = n_b m_b = 100 \cdot 5.24 \times 10^{-16}[kg] = 5.24 \times 10^{-14}[kg] \tag{28}$$

Using eqn 26, and assuming the cell "added mass" is uniformly distributed over the surface of the beam, the change in natural frequency is

$$\Delta\omega \doteq \omega_0(-\frac{1}{2}\frac{\Delta m}{m}) = 1.85 \times 10^5[rad/s] \cdot (-\frac{1}{2}\frac{5.24 \times 10^{-14}[kg]}{1.49 \times 10^{-12}[kg]}) = -3.25 \times 10^3[rad/s] \tag{29}$$

Please be noticed that the frequency, f, in units of $[cycles/sec]$ (or $[Hz]$) is related to the angular frequency (or radian frequency), ω, by $f[Hz] = \omega[rad/s]/(2\pi[rad/cycle])$.

Problem 3 (30 points)

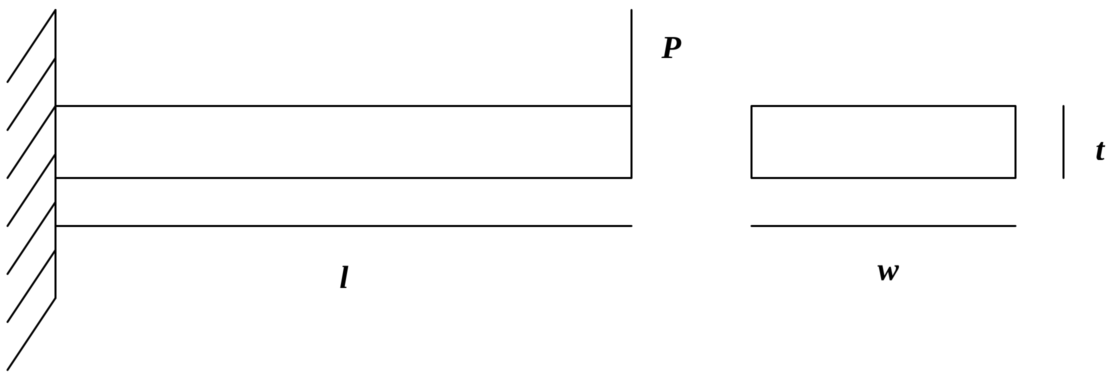
In this problem, the CNT(Carbon nanotube) can be assumed to be a cantilever beam.

For elastic response, the lateral displacement of the cantilever $v(x)$ is related to the bending moment by

$$\frac{d^2v(x)}{dx^2} = \frac{M(x)}{EI} = \frac{P(a-x)}{EI} \tag{30}$$

where a is the distance from $x = 0$ to the loading point. In the following derivation, x will be only in the range of $0 \leq x \leq a$.

Problem 3



Remember that k, the stiffness, is a function of structure parameters such as geometry. That is, for the same material, two beams with different widths for example will have different stiffnesses.

Young's modulus, on the other hand, depends ONLY on the material and thus is a material property. Finding a way to correlate these two is a reasonable approach to solving this problem.

From the definition of stiffness,

$$k = \frac{P}{d} \Leftrightarrow P = k \cdot d \qquad (1)$$

Furthermore, as we learned in class and in the lab, the following is true:

$$d = \frac{P \cdot l^3}{3 \cdot E \cdot I} \Leftrightarrow P = \frac{3 \cdot E \cdot I \cdot d}{l^3} \qquad (2)$$

where $I = \frac{w \cdot t^3}{12} = \frac{50 \cdot 10^{-6}(m) \cdot \left(2.0 \cdot 10^{-6}\right)^3 \left(m^3\right)}{12} = 3.33 \cdot 10^{-23}\left(m^4\right)$

Setting (1) equal to (2) we get:

$$k \cdot d = \frac{3 \cdot E \cdot I \cdot d}{l^3} \Leftrightarrow k = \frac{3 \cdot 107 \cdot 10^9 \left(\frac{N}{m^2}\right) \cdot 50 \cdot 10^{-6}(m) \cdot \left(2 \cdot 10^{-6}\right)^3 \left(m^3\right)}{12 \cdot \left(460 \cdot 10^{-6}\right)^3 \left(m^3\right)} = 0.11\, N/m \qquad (3)$$

From Equation 3, it can be concluded that k is proportional to the width and $(thickness)^3$ of the beam and inversely proportional to the $(length)^3$. Therefore, to get the minimum stiffness, we need to use the minimum width and thickness and the maximum length for our calculations using Equation 3 and vice versa for the maximum stiffness.

More specifically:

$$k_{\min} = \frac{3 \cdot 107 \cdot 10^9\ N/m^2 \cdot 47 \cdot 10^{-6} \cdot (1.5 \cdot 10^{-6})^3}{12 \cdot (465 \cdot 10^{-6})^3} = 0.04\, N/m$$

and

$$k_{\max} = \frac{3 \cdot 107 \cdot 10^9\ N/m^2 \cdot 53 \cdot 10^{-6} \cdot (2.5 \cdot 10^{-6})^3}{12 \cdot (455 \cdot 10^{-6})^3} = 0.24\, N/m$$

Problem 4

Let's derive the necessary results instead of using Table 8.1 in CDL.

For beam (a), using the sign convention found in CDL, we have:

Force balance: $\frac{dV(x)}{dx} + q(x) = 0 \Leftrightarrow \frac{dV(x)}{dx} = -q(x) = -(-w) = w \Leftrightarrow V(x) = w \cdot x + C_1$ (1)

Boundary Condition: $V(L) = 0 \Leftrightarrow C_1 = -w \cdot L$

Thus, $V(x) = w \cdot x - w \cdot L$ (2)

Moment balance:

$$\frac{dM(x)}{dx} + V(x) = 0 \Leftrightarrow \frac{dM(x)}{dx} = w \cdot (L - x) \Leftrightarrow M(x) = -\frac{w \cdot x^2}{2} + w \cdot L \cdot x + C_2 \quad (3)$$

Boundary Condition: $M(L) = 0 \Leftrightarrow C_2 = -\frac{w \cdot L^2}{2}$

Thus, $M(x) = -\frac{w \cdot x^2}{2} + w \cdot L \cdot x - \frac{w \cdot L^2}{2}$ (4)

Converting it to an equation with dimensionless parameters for graphing purposes:

$$\frac{M(x)}{w \cdot L^2} = -\left(\frac{x}{L}\right)^2 + \frac{x}{L} - \frac{1}{2} \tag{5}$$

From Equation 3, we can conclude that $M(x)$ is a strictly increasing function

since $0 \le x \le L$. Thus, $M_{\min} = M(0) = -\frac{w \cdot L^2}{2}$ (6)

From the moment-curvature relation and Equation 4:

$$\frac{d^2 v(x)}{dx^2} = \frac{M}{E \cdot I} \Leftrightarrow \frac{dv(x)}{dx} = \frac{1}{E \cdot I} \cdot \left(-\frac{w \cdot x^3}{6} + \frac{w \cdot L \cdot x^2}{2} - \frac{w \cdot L^2 \cdot x}{2}\right) + C_3 \tag{7}$$

Boundary Condition: $\left.\frac{dv(x)}{dx}\right|_{x=0} = 0 \Leftrightarrow C_3 = 0$

Thus,

$$\begin{aligned} &\frac{dv(x)}{dx} = \frac{1}{E \cdot I} \cdot \left(-\frac{w \cdot x^3}{6} + \frac{w \cdot L \cdot x^2}{2} - \frac{w \cdot L^2 \cdot x}{2}\right) \Leftrightarrow \\ &\Leftrightarrow v(x) = \frac{1}{E \cdot I} \cdot \left(-\frac{w \cdot x^4}{24} + \frac{w \cdot L \cdot x^3}{6} - \frac{w \cdot L^2 \cdot x^2}{4}\right) + C_4 \end{aligned} \tag{8}$$

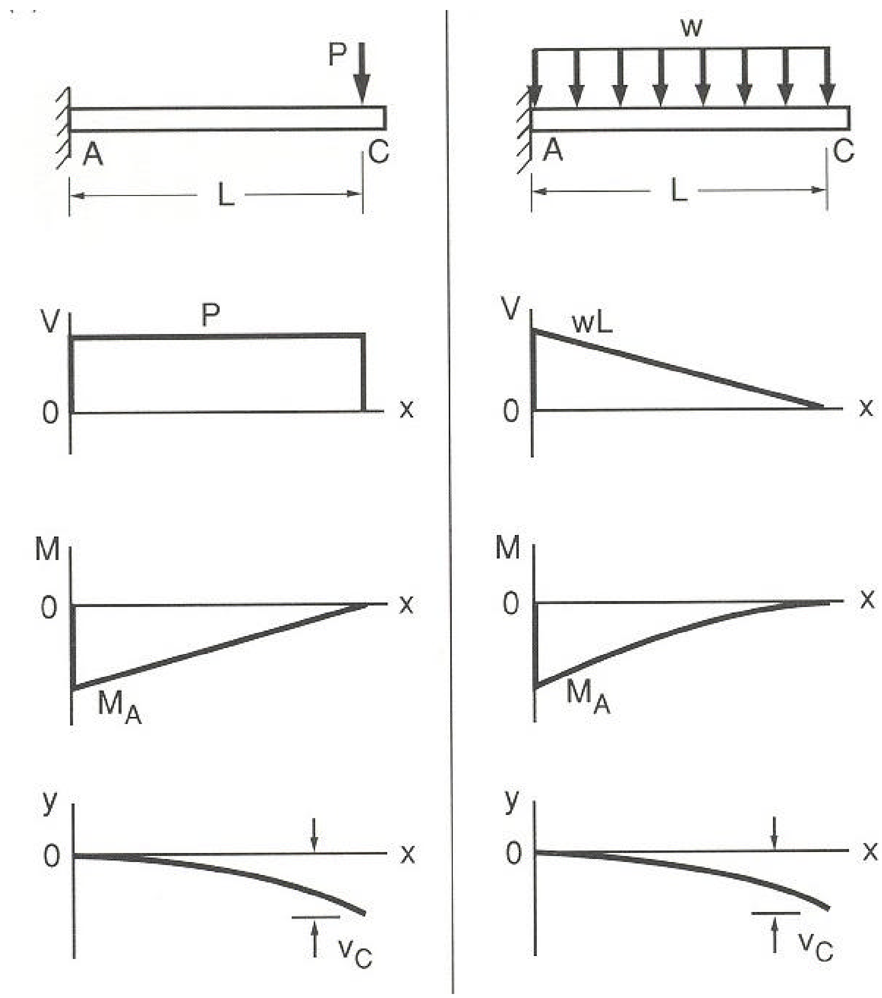
Boundary Condition: $v(0) = 0 \Leftrightarrow C_4 = 0$

Thus, $d(x) = -v(x) = \frac{w}{24 \cdot E \cdot I} \cdot \left(x^4 + 6 \cdot L^2 \cdot x^2 - 4 \cdot L \cdot x^3\right)$ and $d_{\max} = \frac{w \cdot L^4}{8 \cdot E \cdot I}$ (9)

Again, normalizing this equation for graphing purposes yields

$$\frac{E \cdot I \cdot d(x)}{w \cdot L^4} = \frac{1}{24} \cdot \left(\frac{x}{L}\right)^4 - \frac{1}{6} \cdot \left(\frac{x}{L}\right)^3 + \frac{1}{4} \cdot \left(\frac{x}{L}\right)^2 \tag{10}$$

The following figure summarizes the deflection and moment profile along the beam



Let's use the principle of superposition for part (b)

This problem can be broken into two parts: a beam with a uniformly distributed load and an identical beam with an unknown tip load R_L. We know the deflection d_1 and bending moment M_1 for the first part from part (a) (Equations 4 & 8) and for the second part (d_2 and M_2 accordingly) from the lab handout:

$$M(x) = R_L \cdot (L - x) \tag{9}$$

$$d(x) = -\frac{R_L \cdot x^2 \cdot (3 \cdot L - x)}{6 \cdot E \cdot I} \tag{10}$$

Make sure you use consistent sign convention. The assumption here is that this is an upwards-directed tip load.

By superposition:

$$d_{total} = d_1 + d_2 = \frac{w \cdot x^2}{24 \cdot E \cdot I} \cdot \left(x^2 + 6 \cdot L^2 - 4 \cdot L \cdot x\right) - \frac{R_L \cdot x^2 \cdot (3 \cdot L - x)}{6 \cdot E \cdot I}$$

where R_L is unknown. Using $d_{total}\big|_{x=L} = 0$ in the equation above we get

$R_L = \frac{3 \cdot w \cdot L}{8}$ and total displacement becomes:

$$d_{total} = \frac{w \cdot x^2}{48 \cdot E \cdot I}\left(2 \cdot x^2 - 5 \cdot L \cdot x + 3 \cdot L^2\right) \qquad (11)$$

and $d'_{total} = \frac{w}{48 \cdot E \cdot I} \cdot x \cdot \left(8 \cdot x^2 - 15 \cdot L \cdot x + 6 \cdot L^2\right) = 0 \qquad (12)$

which has two roots in $0 \leq x \leq L$: $x = 0$ and $x = \frac{L \cdot \left(15 - \sqrt{33}\right)}{16}$, which is also the point where deflection is maximum $d_{total\_\max} \approx 0.005 \cdot \frac{w \cdot L^4}{E \cdot I}$

Similarly for the bending moment:

$$M_{total} = M_1 + M_2 = -\frac{w \cdot x^2}{2} + w \cdot L \cdot x - \frac{w \cdot L^2}{2} + \frac{3 \cdot w \cdot L}{8} \cdot (L - x) \qquad (13)$$

and $M'_{total} = -w \cdot x + w \cdot L - \frac{3 \cdot w \cdot L}{8} = 0 \Leftrightarrow x = \frac{5 \cdot L}{8}$

Thus, $M_{total\_\max} = \frac{9 \cdot w \cdot L^2}{128}$