

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139
2.002 MECHANICS AND MATERIALS II
HOMEWORK NO. 1

Distributed: Wednesday, February 11, 2004
Due: Wednesday, February 18, 2004

Problem 1 (35 points)

A cantilever beam has been constructed from a steel having Young's modulus $E = 208 \text{ GPa}$, Poisson ratio $\nu = 0.29$, and tensile yield strength $\sigma_y = 410 \text{ MPa}$. The length of the beam, from its base to its tip, is $L = 1 \text{ m}$, and the uniform cross-section is rectangular, $h = 5 \text{ mm}$ thick and $b = 30 \text{ mm}$ wide.

Two axially-mounted strain gages have been mounted on the top surface ($y = h/2$). The precise absolute axial position " x " is not known for either gage; however, it is known that the two gages are spaced an axial distance of $\ell = 200 \text{ mm}$ apart along the length of the beam.

A tip load of magnitude ' P ' is applied to the cantilever, acting in a direction parallel to the y -axis, causing the two gages (#1, which is nearer the base of the cantilever, and #2, which is nearer its tip) to register the following strain values:

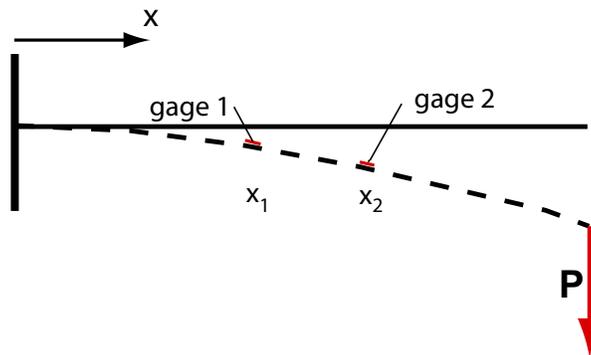


Figure 1: Schematic of tip-loaded cantilever showing positions of strain gages numbers 1 and 2.

Gage no.	axial coordinate	strain
1	$x_1 = ?$	1200×10^{-6}
2	$x_2 = x_1 + 200 \text{ mm}$	900×10^{-6}

- **(20 points)** Evaluate the load P .
- **(15 points)** Using the (now-known) value of P , find the axial locations (x_1 and x_2) of both gages.

Problem 2. (25 points)

There is current interest in the use of microfabricated cantilever beams in detecting the presence of bacteria in a liquid solution. Following fabrication of the cantilever, its surface is coated with an antibody that is specific to the presence of the bacterium of interest, and the free vibration characteristics of the cantilever (its first-mode natural frequency, ω_0) is recorded experimentally.

Then the coated cantilever is exposed to a liquid medium. If the sought-for bacteria are present in solution, they will preferentially attach themselves to the antibody coating on the surface of the cantilever, in the process increasing the vibrating mass of the cantilever by an amount $\Delta m = n_b m_b$, where n_b is the number of bacterium cells that attach, and m_b is the mass of the bacterium cell.

Image removed due to copyright considerations.

Figure 2: Scanning electron microscope images of *E. coli* bacteria attached to various micro-fabricated resonating cantilever beams. (from: Ilic, et al., *Applied Physics Letters*, **77**, #3, 2000, 450-452.

Subsequent testing of the added-mass cantilever should reveal a progressively decreasing natural frequency as more bacterium cells are added.

The cantilever of interest has been fabricated of silicon nitride, having a mass density of $\rho = 3.1 \times 10^3 \text{ kg/m}^3$ and Young's modulus of $E = 100 \text{ GPa}$. It has a uniform rectangular cross section, of thickness $h = 320 \text{ nm}$, width $b = 15 \text{ }\mu\text{m}$, and total length $L = 100 \text{ }\mu\text{m}$.

- Assuming that the antibody coating itself does not appreciably affect the natural frequency of the cantilever, estimate the first-mode natural frequency, ω_0 , of the cantilever

in the absence of adhered bacteria. (Note: please refer to the Lab 1 Handout Notes on vibration of cantilever beams for relevant analysis.)

- Assume that the added mass $\Delta m = n_b m_b$ is uniformly distributed along the length of the beam, and further, assume that the presence of the adhered bacteria does not affect the stiffness of the beam. Let the total mass of the coated beam be $m = \rho b h L$; assuming that $\Delta m \ll m$, show that the change in frequency resulting from the added mass, $\Delta\omega$, can be expressed by

$$\omega_0 + \Delta\omega \doteq \omega_0 \left(1 - \frac{1}{2} \frac{\Delta m}{m} \right).$$

- Hint # 1: You might wish to show that the natural frequency of a uniform beam of total mass ‘ m ’ can be written as $\omega \propto (1/L)^2 \sqrt{EIL/m}$ (neglecting dimensionless factors).
 - Hint # 2: If the change in total mass (Δm) is very small in comparison to the initial beam mass (m), then the resulting frequency of the perturbed-mass system can be evaluated by taking a Taylor series expansion of the frequency expression about the reference mass.
- Assume that the cantilever has been covered by $n_b = 100$ bacterium cells. From the figures, it appears that the diameter, D_b , of each bacterium cell is near $D_b = 1\mu m$; assuming each cell to be spherical and to have a mass density, ρ_b , equal to that of water, justify an estimate of each cell’s mass as $m_b \doteq 5.24 \times 10^{-16} kg$. Using this estimate, and assuming that the cell “added mass” is uniformly distributed over the surface of the beam, evaluate the change in natural frequency that you can expect to see for the bacterium-coated cantilever.

Problem 3. (20 points)

Atomic force microscopy resolves the structure of surfaces to near atomic-level resolution. A key structural element of an atomic force microscope (AFM) is a small cantilevered beam fabricated from a material such as silicon using lithographic technology (e.g., precision etching). The beam has a sharp-tipped diamond stylus at its tip, the stylus is tracked across the specimen surface, and very local surface interactions between the tip and the specimen surface cause the tip of the cantilever to deflect. The magnitude of the tip deflection is measured by reflecting a laser beam off its back side and imaging the location of the reflected light. The tip/surface forces are computed from the deflections using beam theory, and they are used to map the structure of the surface.

Figure 3 shows a small rectangular cross-section cantilever from an atomic force microscope (AFM) that has been machined from a single block of silicon using lithographic technology (e.g., precision etching). The nominal dimensions of this cantilever are length $\ell = 460 \mu\text{m}$, width $w = 50 \mu\text{m}$, and thickness $t = 2.0 \mu\text{m}$. Elastic constants of silicon are Young's modulus $E = 107 \text{ GPa}$, and Poisson ratio $\nu = 0.2$.

Image removed due to copyright considerations.

The stiffness $k = P/\delta$ of the AFM cantilever is an important design parameter.

- **(10 points)** Compute the stiffness of this AFM cantilever.
- **(10 points)** Unfortunately, dimensional tolerances in the etching technique inevitably lead to a range of dimensions over a population of the AFM cantilever products. The manufacturer quotes the following tolerances for this cantilever: $\ell = 460 \mu\text{m} \pm 5 \mu\text{m}$; $w = 50 \mu\text{m} \pm 3 \mu\text{m}$; and $t_{\min} = 1.5 \mu\text{m} \leq t \leq t_{\max} = 2.5 \mu\text{m}$. Accordingly, the stiffness of any given cantilever can vary from its nominal value, depending on its actual dimensions. **Provide estimates of the minimum and maximum stiffnesses**, k_{\min} and k_{\max} , that can be expected in this cantilever, corresponding to extreme ranges of the dimensions.

Problem 4 (20 points)

CDL Problem 8.15

Hint/suggestion: Problem 8.15a is the continuation of the “self-weight” cantilever problem we looked at in the lecture. You should be well on the way to solving for its moment distribution, and then you should be able to follow the procedures adopted for the tip-loaded cantilever to calculate the corresponding lateral displacement distribution, $v(x)$, and anything else needed.

Problem 8.15(b) is statically indeterminate, meaning that there is no direct way to evaluate its bending moment distribution, $M(x)$, without simultaneously considering the displacement distribution, $v(x)$. There are two straightforward ways to approach such statically indeterminate beam problems.

- Combine CDL equations 3.11 and 3.12 to obtain

$$M''(x) - q(x) = 0 \Rightarrow M''(x) + w = 0,$$

and then use $M(x) = EIv''(x)$ to obtain

$$v''''(x) + \frac{w}{EI} = 0.$$

From here on, there are 4 integrations, 4 constants of integration, and 4 boundary conditions (on $v(0)$, $\phi(0)$, $M(L)$, and $v(L)$) to determine the constants. It will work, but the algebra is deadly dull.

- On the other hand, you can use the principle of superposition. In this case, observe that the stated problem is equivalent to a cantilever, subject to (a) the distributed loading $q(x) = -w$, **AND** to a concentrated upwards-directed tip load, of [unknown] magnitude R_L . The total vertical tip displacement of the cantilever is $v_{\text{total}}(L) = 0$ and by superposition,

$$0 = v_{\text{total}}(L) = v_{R_L}(L) + v_q(L),$$

where it is understood that there is a downward tip displacement, $v_q(L) < 0$ (from problem 8.15a), as well as an opposite-direction tip displacement, $v_{R_L} > 0$, from the unknown positive-upward tip reaction force, R_L . This leads to a simple linear equation to solve for R_L ; once R_L is obtained, all parts of the solution to 8.15b are known by superposition... Try whichever method suits you.