2.001 - MECHANICS AND MATERIALS I
Lecture #26
12/11/2006
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Energy Methods

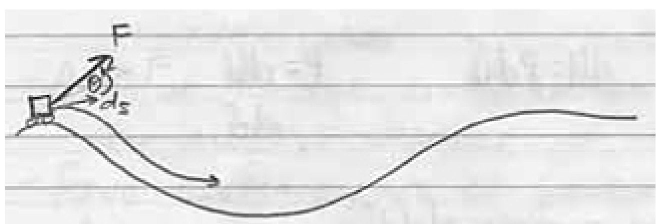
3 Basic Ingredients of Mechanics:

1. Equilibrium
2. Constitutive Relations
   $\sigma - \epsilon \ldots$ Stress-Strain
   $F - \delta$ Force-Deformation
3. Compatibility
   Dealing with geometric considerations

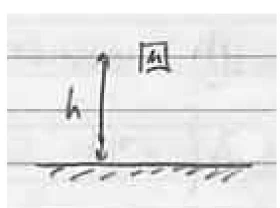
Castigliano's Theorem

Start with work:

$$W = \int \vec{F} \cdot d\vec{s}$$



Conservative Forces:
EX: Gravity

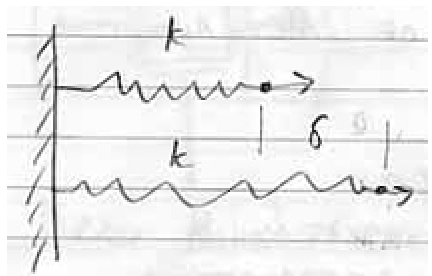


$$u = mgh \Rightarrow \text{ Potential Energy, Path Independent}$$

$$W = \int \vec{F} \cdot d\vec{s} = \int_0^h mgdz = mgh$$

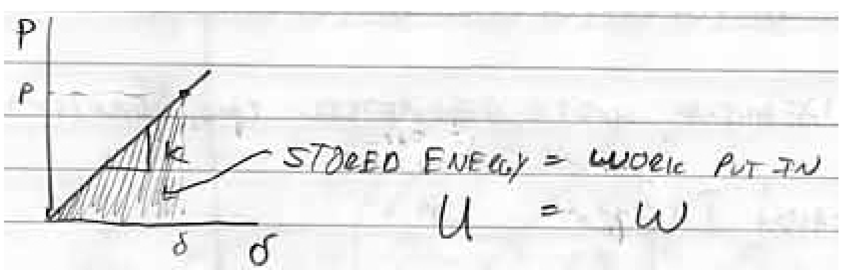
Note: Elastic systems are conservative.
Plastic deformation is *not* conservative.

EX: Springs



Energy $= 1/2k\delta^2$

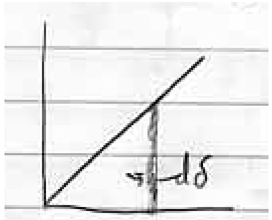
$$W = \int_0^\delta Pd\delta = \int_0^\delta k\delta d\delta = \frac{1}{2}k\delta^2$$



Stored Energy = Work Put In

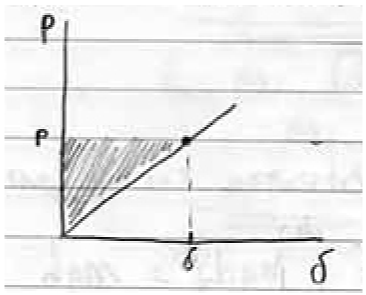
$$U = W$$

What if the spring were stretched a little further?



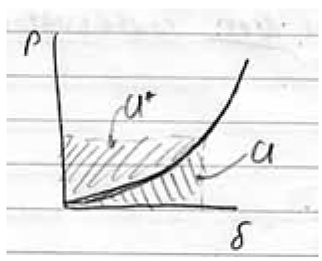
$$dU = Pd\delta$$
$$P = \frac{dU}{d\delta}$$

Complementary Energy (U^*)



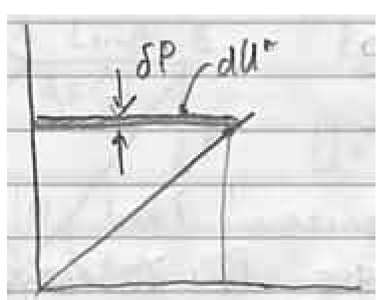
$$U^* = \int \delta dP$$

Note: $U = U^*$ due to linearity



$$U \neq U^* \text{ if this is non-linear}$$

What if the load were stretched a little further?



$$dU^* = \delta dP$$
$$\delta = \frac{dU^*}{dP}$$

Recall, for linear elastic material

$$\delta = \frac{dU}{dP}$$

Castigliano's Theorem:

Express complementary energy in terms of the loads

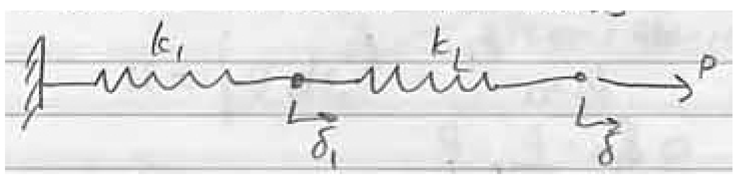
Add up all the complementary energy stroed in all the pieces

Find displacement of given point from derivative of U^* with respect to P at that point in that direction

Example: Linear Spring

$$U^* = \int \delta dP = \int \frac{P}{k} dP = \frac{P^2}{2k} = \frac{k^2\delta^2}{2k} = \frac{1}{2}k\delta^2$$

Example: Springs in series



$$F_1 = F_2 = P$$

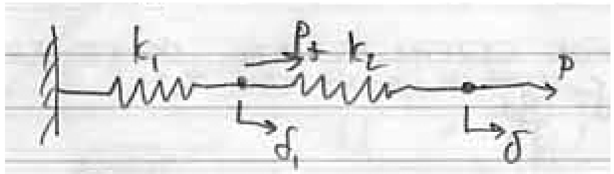
$$U = \sum_i U_i = \frac{F_1^2}{2k_1} + \frac{F_2^2}{2k_2} = \frac{P^2}{2k_1} + \frac{P^2}{2k_2}$$

Use Castigliano's Theorem

$$\delta = \frac{U^*}{dP} = \frac{P}{k_1} + \frac{P}{k_2}$$

Find δ_1 using Castigliano's Theorem

Put a fictitous P_f to coinside with δ_1



$$\sum F_x = 0$$

$$-F_1 + P_f + P = 0 \Rightarrow F_1 = P_f + P$$

$$-F_2 + P = 0 \Rightarrow F_2 = P$$

$$U = \frac{F_1^2}{2k_1} + \frac{F_2^2}{2k_2} = \frac{(P_f + P)^2}{2k_1} + \frac{P^2}{2k_2}$$

$$\delta_1 = \frac{\partial U}{\partial P_f} = \frac{P_f + P}{k_1}$$

Recall P_f =0, so:

$$\delta_1 = \frac{P}{k_1}$$

$$\delta = \frac{dU}{dP} = \frac{P_f + P}{k_1} + \frac{P}{k_2}$$

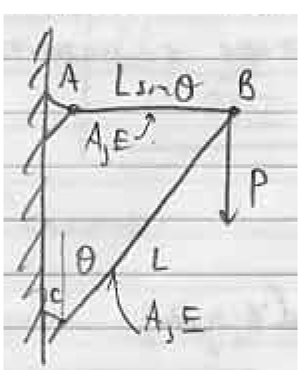
$$P_f = 0$$

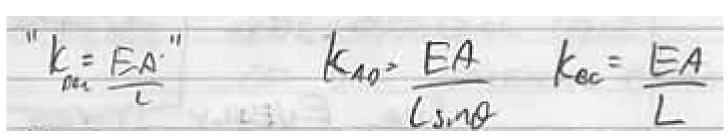
So:

$$\delta = \frac{P}{k_1} + \frac{P}{k_2}$$

Note: This was done without doing compatibility explicitly.

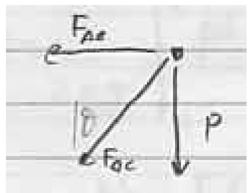
Example Truss





$$U = \sum_i U_i = \frac{F_{AB}^2}{2k_{AB}} + \frac{F_{BC}^2}{2k_{BC}}$$

Equilibrium of B



$$\sum F_x = 0$$
$$F_{AB} - F_{BC} \sin\theta = 0$$

$$\sum F_y = 0$$
$$-P - F_{BC} \cos\theta = 0$$
$$P = -F_{BC} \cos\theta$$
$$F_{BC} = \frac{-P}{\cos\theta}$$
$$F_{AB} = P \tan\theta$$

$$U = \frac{(P\tan\theta)^2}{2k_{AB}} + \left(\frac{P}{\cos\theta}\right)^2 \frac{1}{2k_{BC}}$$
$$-u_y^B = \frac{dU}{dP} = \frac{P\tan^2\theta}{k_{AB}} + \frac{P}{\cos^2\theta k_{BC}}$$

Strain Energy Density

$$W = \int F d\delta$$
$$u = \int \sigma d\epsilon$$

This is useful in beam bending.

Stored Energy in A Beaam $(\sigma_{xx}, \epsilon_{xx})$

Total Energy Stored

$$U = \int_V dV \int_0^{\epsilon} \sigma_{xx} d\epsilon_{xx}$$

Recall:

For a beam:

$$\sigma_{xx} = E\epsilon_{xx}$$

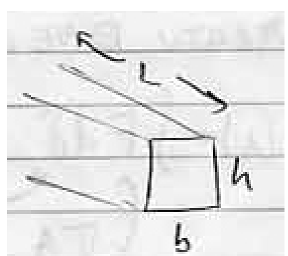
For one material:

$$\sigma_{xx} = \frac{-My}{I}$$

Beam:

$$\begin{aligned} U &= \int_V dV \int_{\epsilon} \sigma_{xx} d\epsilon_{xx} \\ &= \int_V dV \int_0^{\sigma_{xx}} \frac{\sigma_{xx} d\sigma_{xx}}{E} \\ &= \int_V dV \frac{\sigma_{xx}^2}{2E} \end{aligned}$$

For one material:

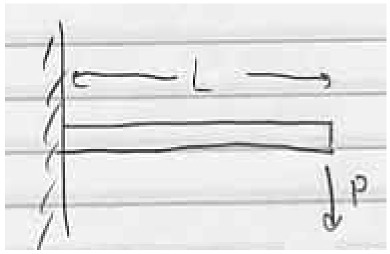


$$U = \int_V \frac{M(x)^2}{2EI^2} y^2 dV$$

So:

$$U = \frac{1}{2EI} \int_a M(x)^2 dx \text{ For special case of one material beam.}$$

Example:



$$M(x) = -P(L - x)$$

$$U = \frac{1}{2EI} \int_0^L P^2 (L - x)^2 dx = \frac{P^2 L^3}{6EI}$$

$$\delta_{tip} = \frac{dU}{dP} = \frac{PL^3}{3EI}$$