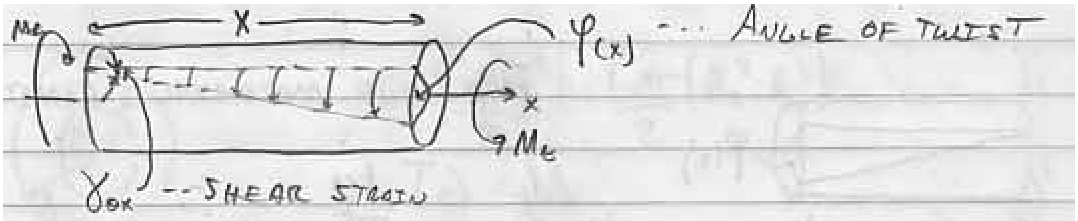
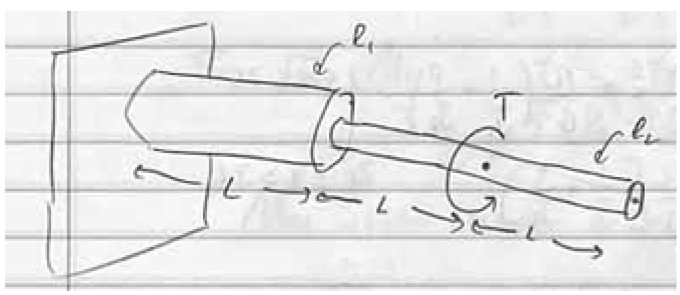
2.001 - MECHANICS AND MATERIALS I
Lecture #25
12/6/2006
Prof. Carol Livermore

Recall from last time:



	Beam Bending	Shaft Torsion
Deformation	$\frac{1}{\rho} = k = \frac{d\theta}{dx}$	$\frac{d\varphi}{dx}$
Strain (Compatibility)	$\epsilon_{xx} = \frac{-y}{\rho}$	$\epsilon_{\theta x} = \frac{r}{2}\frac{d\varphi}{dx}$
Equilibrium	$M_z = \int_A \sigma_{xx} y dA$	$M_t = \int_A \sigma_{\theta x} r dA$
Stress (Constitutive)	$\sigma_{xx} = \frac{-Ey}{\rho}$	$\sigma_{\theta\theta} = Gr\frac{d\varphi}{dx}$
Moment-Deformation	$M_z = \frac{EI}{\rho}$	$M_t = GJ\frac{d\varphi}{dx}$
Moment of Inertia	$I = \int_A y^2 dA$	$J = \int_A r^2 dA$
Stress-Loading	$\sigma_{xx} = \frac{-M_z y}{I}$	$\sigma_{\theta x} = \frac{M_t r}{J}$
Deflections/Angle of Twist	$\frac{d\theta}{dx} = \frac{1}{\rho} = \frac{M_z}{EI}$	$\frac{d\varphi}{dx} = \frac{M_t}{GJ}$

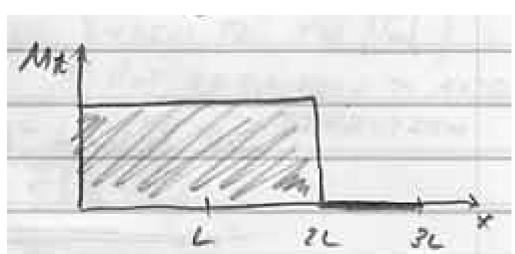
Example:



What is $\varphi(3L)$?

$$M_t = T, 0 \leq x \leq 2L$$

$$M_t = 0, 0 \leq x \leq 3L$$



$$M_t = GJ\frac{d\varphi}{dx}$$
$$J = \frac{\pi}{2}R_1^4 = J_1, 0 \le x \le L$$
$$J = \frac{\pi}{2}R_2^4 = J_2, L \le x \le 3L$$
$$\frac{d\varphi}{dx} = \frac{T}{GJ_1}, 0 \le x \le L$$
$$\frac{d\varphi}{dx} = \frac{T}{GJ_2}, L \le x \le 2L$$
$$\frac{d\varphi}{dx} = 0, 2L \le x \le 3L$$

So:

$$\varphi(x) = \frac{T}{GJ_1}x + c_1, 0 \le x \le L$$
$$\varphi(x) = \frac{T}{GJ_2}x + c_2, L \le x \le 2L$$
$$\varphi(x) = c_3, 2L \le x \le 3L$$

Boundary Conditions:

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi(L)$ is continuous $\frac{TL}{GJ_1} = \frac{TL}{GJ_2} + c_2 \Rightarrow c_2 = \frac{TL}{G}\left(\frac{1}{J_1} - \frac{1}{J_2}\right)$

$\varphi(2L)$ is continuous. $\frac{2TL}{GJ_2} + \frac{TL}{G}\left(\frac{1}{J_1} - \frac{1}{J_2}\right) = c_3 \Rightarrow c_3 = \frac{TL}{G}\left[\frac{1}{J_1} + \frac{1}{J_2}\right]$

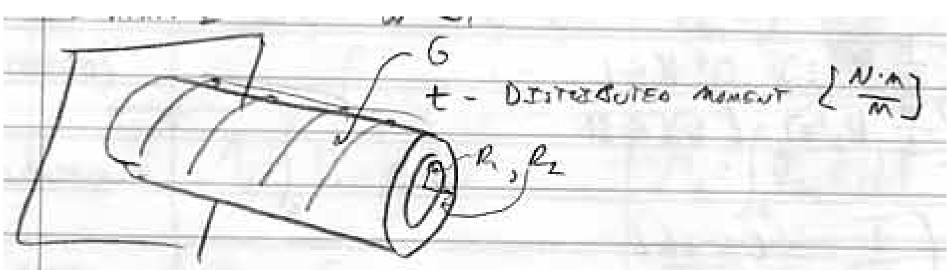
Thus:

$$\begin{aligned}\varphi(x) &= \frac{T}{GJ_1}, 0 \le x \le L \\ &= \frac{T}{GJ_2}x + \frac{TL}{G}\left(\frac{1}{J_1} - \frac{1}{J_2}\right) L \le x \le 2L \\ &= \frac{TL}{G}\left[\frac{1}{J_1} + \frac{1}{J_2}\right], 2L \le x \le 3L\end{aligned}$$

So:

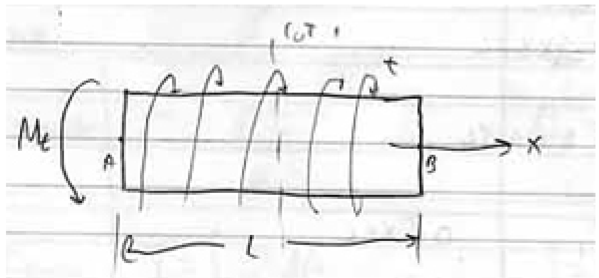
$$\varphi(3L) = \frac{TL}{G}\left[\frac{1}{J_1} + \frac{1}{J_2}\right]$$

Example: Hollow Shaft



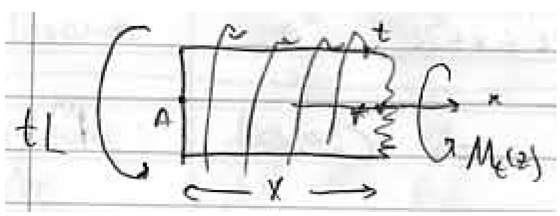
Q: Max stress? Angle of twist?

FBD

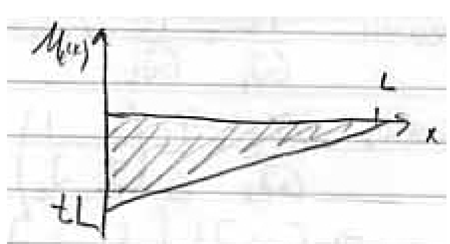


$$\sum M_x = 0$$
$$M_t - \int_0^L t dx = 0$$
$$M_t = tL$$

FBD Cut 1



$$\sum M_x = 0$$
$$tL - tx + M_z(x) = 0$$
$$M_t(z) = t(x - L)$$



$$J_{hollowshaft} = \frac{\pi}{2}\Bigg(R_2^4 - R_1^4)$$

So:

$$\sigma_{\theta x} = \frac{M_t r}{J} = \frac{-t(L-x)r}{\frac{\pi}{2}(R_2^4 - R_1^4)}$$

Max shear stress

$$\sigma_{\theta x_{max}} = \frac{M_t(0)R_2}{J} = \frac{-tLR_2}{\frac{\pi}{2}(R_2^4 - R_1^4)}$$

Angle of Twist

$$\frac{d\varphi}{dx} = \frac{M_t}{GJ}$$

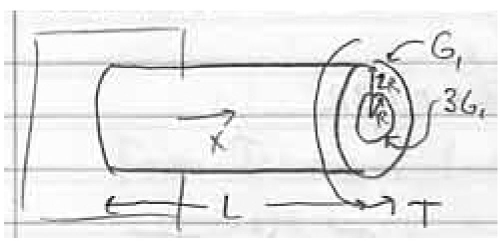
$$\frac{d\varphi}{dx} = \frac{-t(L-x)}{GJ}$$

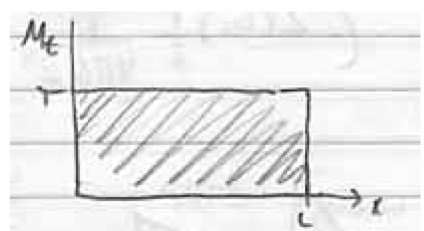
So:

$$\varphi(x) = \frac{-t}{GJ}\left[Lx - \frac{x^2}{2}\right]$$

$$\varphi(L) = \frac{-t}{GJ}\frac{L^2}{2}$$

Example:





Find: $\sigma_{\theta x_{max}}$

$$\begin{aligned}
M_t &= \frac{d\varphi}{dx}\int_A G(r)r^2 dA \\
&= \frac{d\varphi}{dx}\left[\int_0^{2\pi}\int_0^{R_1} 3G_1 r^2 r dr d\theta + \int_0^{2\pi}\int_{R_1}^{2R_1} G_1 r^2 r dr d\theta\right] \\
&= 2\pi\left(\left[3G_1\frac{r^4}{4}\right]_0^{R_1} + \left[G_1\frac{r^4}{4}\right]_{R_1}^{2R_1}\right) \\
&= \frac{d\varphi}{dx}2\pi\left[\frac{3}{4}G_1 R^4 + \frac{G_1}{4}\Big(16R^4 - R^4\Big)\right]
\end{aligned}$$

$$T = \frac{d\varphi}{dx}9\pi G_1 R^4$$

Note: Could have $(GJ)_{eff} = \sum_i G_i J_i$

So:

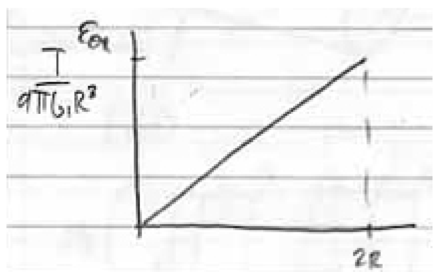
$$\frac{d\varphi}{dx} = \frac{T}{9\pi G_1 R^4}$$

And:

$$\epsilon_{\theta x} = \frac{r}{2}\frac{T}{9\pi G_1 R^4}$$

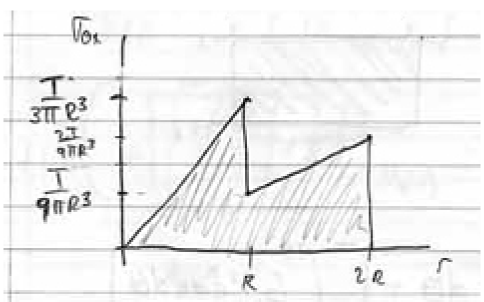
Recall:

$$\sigma_{\theta x} = 2G(r)\epsilon_{\theta x}$$



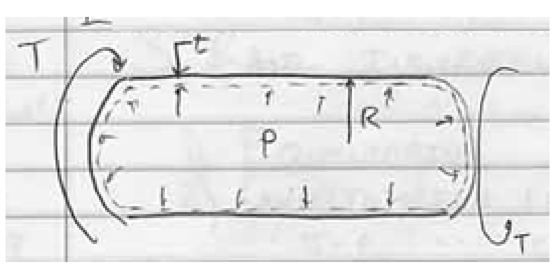
$$\sigma_{\theta x} = 2(3G_1)\frac{r}{2}\frac{T}{9\pi G_1 R^4}, 0 < r < R$$

$$\sigma_{\theta x} = 2(G_1)\frac{r}{2}\frac{T}{9\pi G_1 R^4}, R < r < 2R$$



$$\sigma_{\theta x_{max}} = \sigma_{\theta x}(R) = \frac{T}{3\pi R^3}$$

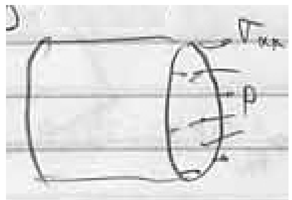
Example: Superposition



Q: What are the principal stresses? Principal directions?
Linearity:

$$\sigma(\text{Load } 1 + \text{Load } 2) = \sigma(\text{Load } 1) + \sigma(\text{Load } 2)$$

Recall Pressure Vessels:
FBD of Cut:

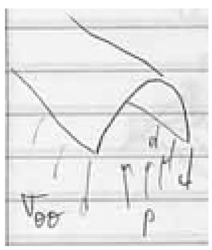


$$\sum F_x = 0$$

$$-P(\pi R^2) + \sigma_{xx}(2\pi Rt) = 0$$

$$\sigma_{xx} = \frac{PR}{2t}$$

FBD of Cut:



$$\sum F_y = 0$$

$$P(2R)(L) - \sigma_{\theta\theta}(2t)(L) = 0$$

$$\sigma_{\theta\theta} = \frac{PR}{t}$$

So:

$$[\sigma]_p = \begin{vmatrix} 0 & 0 & 0 \\ 0 & \frac{PR}{t} & 0 \\ 0 & 0 & \frac{PR}{2t} \end{vmatrix}$$

For torsional load (1 material):

$$\sigma_{\theta x} = \frac{M_t r}{J} = \frac{TR}{J_{hollow}}$$

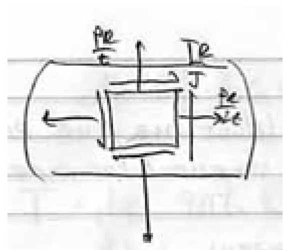
Note: J_{hollow} was calculated earlier.

$$[\sigma]_T = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{TR}{J} \\ 0 & \frac{TR}{J} & 0 \end{vmatrix}$$

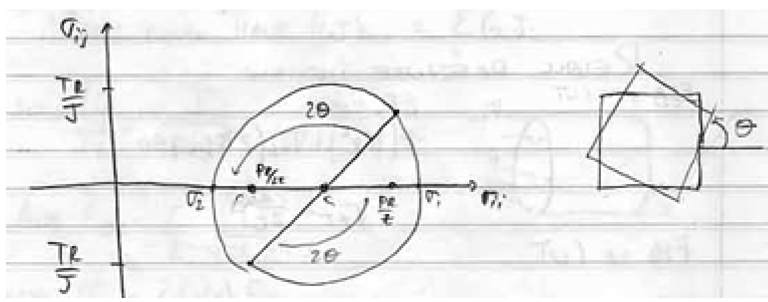
$$J = 2\pi\left(\frac{(2+t)^4}{4} - \frac{R^4}{4}\right) \approx 2\pi R^3 t \text{ throw out higher order terms}$$

$$[\sigma] = [\sigma]_T + [\sigma]_P = \begin{vmatrix} 0 & 0 & 0 \\ 0 & \frac{pR}{T} & \frac{TR}{J} \\ 0 & \frac{TR}{J} & \frac{pR}{2T} \end{vmatrix}$$

Note: This is plane stress.



Draw Mohr's Circle



$$C = \left(\frac{3pR}{4t}, 0\right)$$

$$R = \sqrt{\left(\frac{pR}{4t}\right)^2 + \left(\frac{TR}{J}\right)^2}$$

$$\sigma_{1,2} = \frac{3pR}{4T} \pm \sqrt{\left(\frac{pR}{4t}\right)^2 + \left(\frac{TR}{J}\right)^2}$$

$$\tan\theta = \frac{\left(\frac{TR}{J}\right)}{\left(\frac{pR}{4T}\right)}$$