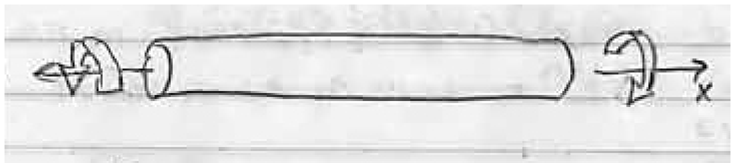
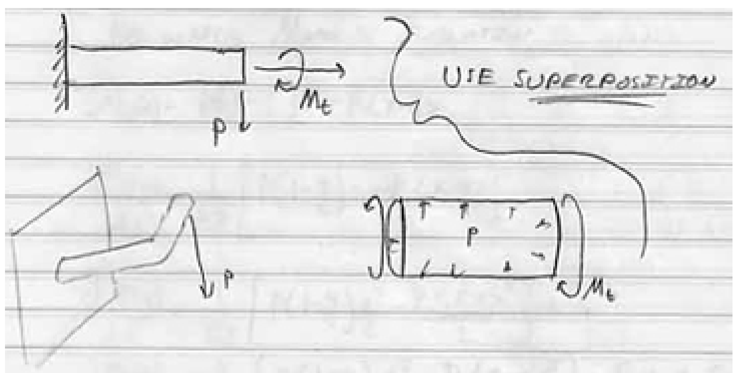
2.001 - MECHANICS AND MATERIALS I
Lecture #24
12/4/2006
Prof. Carol Livermore

Torsion (Twisting)



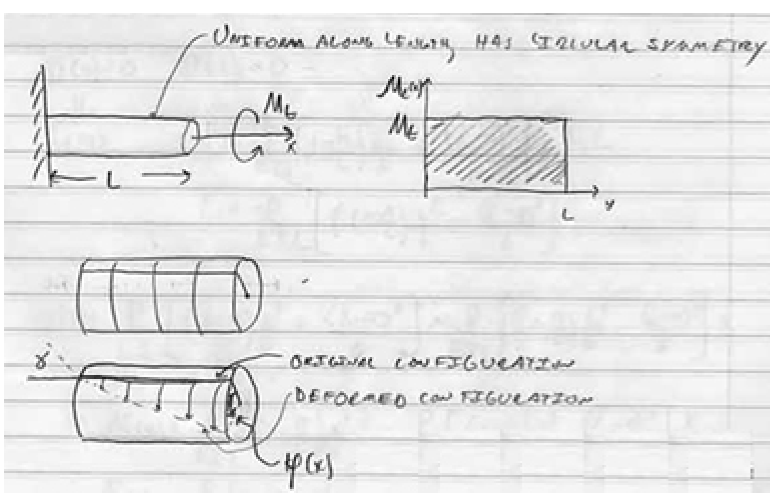
$$M_{xx} = M_t$$

Examples where torsion is important:
- Screwdriver
- Drills
- Propellers

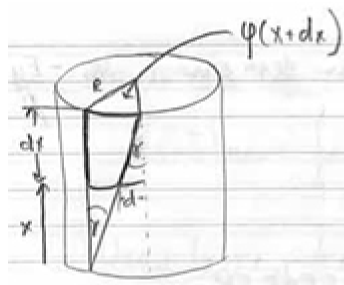


Use superposition

Example: Unifrom along length, has circular symmetry



$$\varphi(x) = \text{Angle of Twist (Total) at Point x (Built Up)}$$



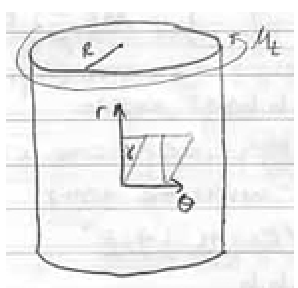
Compatibility:

$$d = \varphi(x + dx)R - \varphi(x)R$$
$$d = \gamma dx$$

So:

$$\gamma dx = \varphi(x + dx)R - \varphi(x)R$$
$$\gamma dx = d\varphi R$$
$$\gamma = R\frac{d\gamma}{dx}$$

What is γ?



So:

$$\gamma = \gamma_{r\theta} \text{ Shear Strain}$$

Thus:

$$\gamma_{r\theta} = R\frac{d\varphi}{dx}$$

And:

$$\epsilon_{\theta x} = \frac{\gamma}{2} = \frac{R}{2}\frac{d\varphi}{dx}$$

$$\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_{xx} = \epsilon_{r\theta} = \epsilon_{rx} = 0$$

Constitutive Relations:

$$\sigma_{\theta x} = 2G\epsilon_{\theta x} = G\gamma_{x\theta} = Gr\frac{d\varphi}{dx}, \text{ where } G \text{ is the shear modulus.}$$

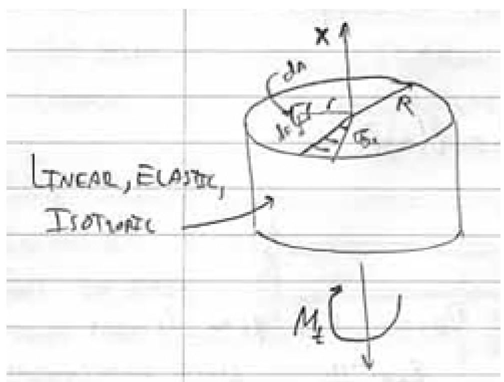
$$G = \frac{E}{2(1+\nu)}$$

$$\sigma_{xx} = \sigma_{\theta\theta} = \sigma_{rr} = \sigma_{r\theta} = \sigma_{rx} = 0$$

$$\sigma_{\theta x} = Gr\frac{d\varphi}{dx}$$

Recall beam bending $\sigma_{xx} = \frac{-Ey}{\rho}$.

Equilibrium:



$$\sum M_x = 0$$

$$-M_t + \int_A r dF = 0$$

$$dF = \sigma_{\theta x} dA$$

$$-M_t + \int_A r\sigma_{\theta x} dA = 0$$

$$M_t = \int_A rGr\frac{d\varphi}{dx} dA$$

$$M_t = \frac{d\varphi}{dx}\int_A Gr^2 dA$$

Recall beam bending:

$$M = \frac{1}{\rho}\int_A Ey^2 dA$$

Note:
$\frac{d\varphi}{dx}$: what happens
$\int_A Gr^2 dA$: effective stiffness
M_t: what we apply

If G is constant "Special Case"

$$M_t = \frac{d\varphi}{dx} G \int_A r^2 dA$$

$$J \equiv \int_A r^2 dA \text{ Polar Moment of Inertia}$$

So for "special case" G is constant.

$$M_t = GJ\frac{d\varphi}{dx}$$

Recall beam bending:

$$M = \frac{EI}{\rho}$$

If G is *not* constant:

$$(GJ)_{eff} = \int_A Gr^2 dA$$

When G is constant:

$$M_t = GJ\frac{d\varphi}{dx} \text{ and generally } \sigma_{\theta x} = Gr\frac{d\varphi}{dx}$$

$$\frac{M_t}{GJ} = \frac{\sigma_{\theta x}}{Gr}$$

So:

$$\sigma_{\theta x} = \frac{M_t r}{J}$$

Example for J

1. Cicular solid shaft

$$J = \int_A r^2 dA = \int_0^{2\pi} \int_0^R r^2 r dr d\theta$$
$$= \int_0^{2\pi} \frac{R^4}{4} d\theta$$

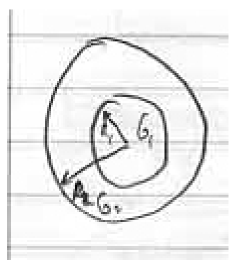
$$J = \frac{\pi}{2} R^4$$

2. Hollow circular shaft

$$J = \int_0^{2\pi} \int_{R_1}^{R_2} r^3 dr d\theta$$

$$J = \frac{\pi}{2}\left(R_2^4 - R_1^4\right)$$

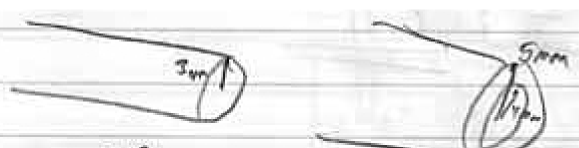
$$(GJ)_{composite} = (GJ)_{inside} + (GJ)_{outside}$$
$$= \frac{G_1 \pi R_1^4}{2} + \frac{G_1 \pi}{2}\left[R_2^4 - R_1^4\right]$$



Or:

$$(GJ)_{eff} = \int_0^{2\pi} \left[\int_0^{R_1} G_1 r^3 dr + \int_{R_1}^{R_2} G_2 r^3 dr\right] d\theta$$

Note: Due to powers of order 4, material buys you more stiffness on the outside than the inside.



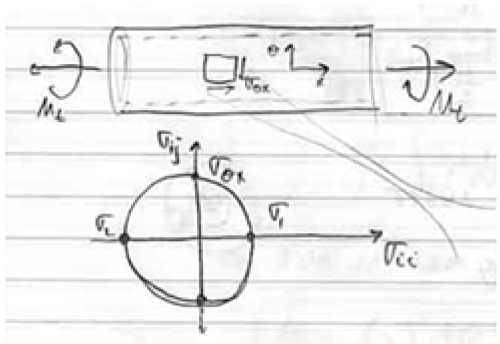
$A = \pi(3)^2$
$= 9\pi \, mm^2$

$A = \pi(5^2 - 4^2)$
$= 9\pi \, mm^2$

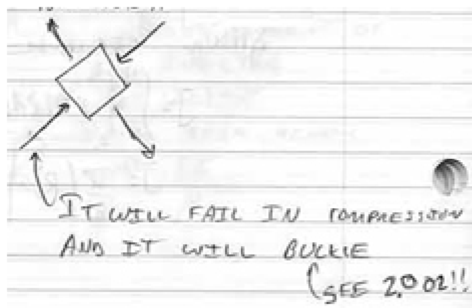
$J = \frac{\pi}{2} 81 \, mm^4$

$J = \frac{\pi}{2}\left(5^4 - 4^4\right) = \frac{\pi}{2} 369 \, mm^4$

So why don't we make R huge?

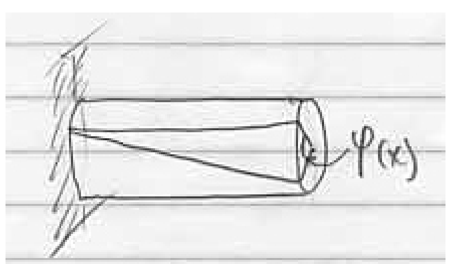


Now look at it at 45°



It will fail in compression and it will buckle.

What is the total angle of twist?



For one material (constant G)

$$M_t = GJ\frac{d\varphi}{dx}$$

So:

$$\frac{d\varphi}{dx} = \frac{M_t}{GJ}$$

$$\varphi(x) = \frac{M_t}{GJ}x + c_1$$

Boundary Condition:

At $x = 0$, $\varphi(0) = 0 \Rightarrow c_1 = 0$.

So:

$$\varphi(x) = \frac{M_t}{GJ}x$$

The total angle of twist $\phi(L)$

$$\phi(L) = \frac{M_t L}{GJ}$$