2.001 - MECHANICS AND MATERIALS I
Lecture #22
11/27/2006
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Beam in pure bending
ρ = radius of curvature
$\epsilon_{xx} = \frac{-y}{\rho}$ $\sigma_{xx} = \frac{-Ey}{\rho}$

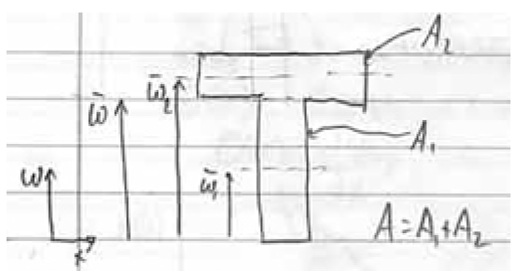
Locating the neutral axis
$\int_A \frac{Ey}{\rho} dA = 0$

Moment-Curvature
$M = \int_A \frac{Ey^2}{\rho} dA$

Special Case: E = constant
Neutral Axis:
$\int_A y dA = 0$

Moment-Curvature
$M = \frac{EI}{\rho}$ $I = \int_A y^2 dA$

Neutral Axis Shortcut
1. Symmetric cross section $\Rightarrow$ neutral axis in the center for E = constant
For E = constant:



Area 1:

$$\int_{A_1} (w - \overline{w_1} dA_1 = 0$$

Area 2:

$$\int_{A_2} (w - \overline{w_2}) dA_2 = 0$$

Total:

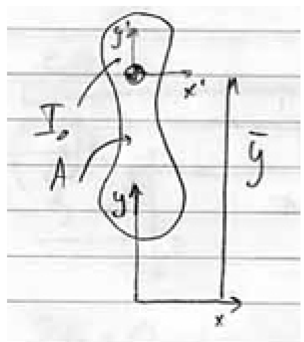
$$\int_{A_1} (w - \overline{w_1} dA_1 + \int_{A_2} (w - \overline{w_2}) dA_2 = 0$$

$$\int_{A_1+A_2} w dA - \overline{w_1}\int_{A_1} dA_1 - \overline{w_2}\int_{A_2} dA_2 = 0$$
$$\overline{w}A = \overline{w_1}A_2 + \overline{w_2}A_2$$
$$\overline{w} = \frac{\sum_i \overline{w_i}A_i}{\sum_i A_i}$$

For a general beam:

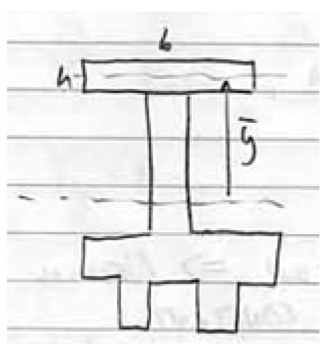
$$\overline{w} = \frac{\sum_i \overline{w_i}A_iE_0}{\sum_i A_iE_i}$$

3. Parallel Axis Theorem:



$$I_{y=0} = I_0 + \overline{y}^2 A$$

EXAMPLE:



$$I_{top} = \frac{bh^3}{12} + \overline{y}^2 bh$$
$$I_{totatl} = I_{top} + I_{mid} + \dots$$

Effective Bending Stiffness $(EI)_{eff}$

$$(EI)_{eff} = \sum_i E_i I_i^{y=0}$$

Beam Deflection (Displacement and slope)

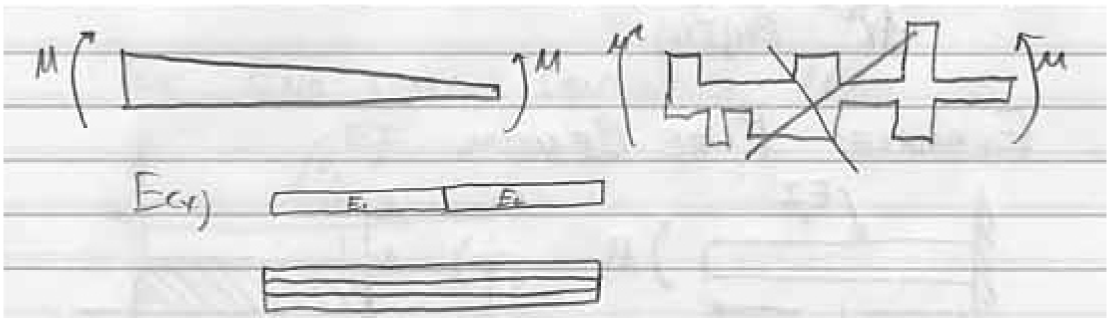
Recall for a one material beam

$$M(x) = EI \frac{1}{\rho(x)}$$

Note for $E \neq$ constant:

$$M(x) = (EI)_{eff} \frac{1}{\rho(x)}$$

Note for gradual change in cross section, this works.

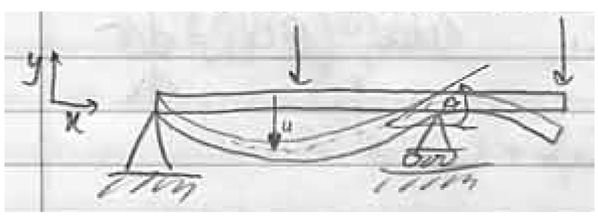


So:

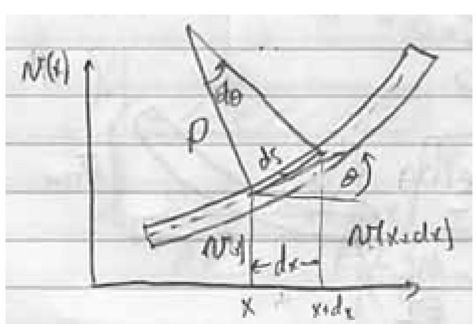
$$M(x) = \frac{E(x)I(x)}{\rho(x)}$$

Example:

Draw what you think will happen.



$u(x)$ - Beam deflection, $\theta(x)$ - beam slope, $\theta(x) = \frac{dv(x)}{dx}$



$ds = \rho d\theta \Rightarrow d\theta = \frac{1}{\rho} ds$, $\theta = \frac{dv}{dx}$, $\frac{d\theta}{dx} = \frac{d^2 v}{dx^2} \Rightarrow d\theta = \frac{d^2 v}{dx^2} dx$

$\cos\theta = \frac{ds}{dx} \approx 1$ so $ds \approx dx$.
So:

$$\frac{1}{\rho} = \frac{d^2 v}{dx^2}$$

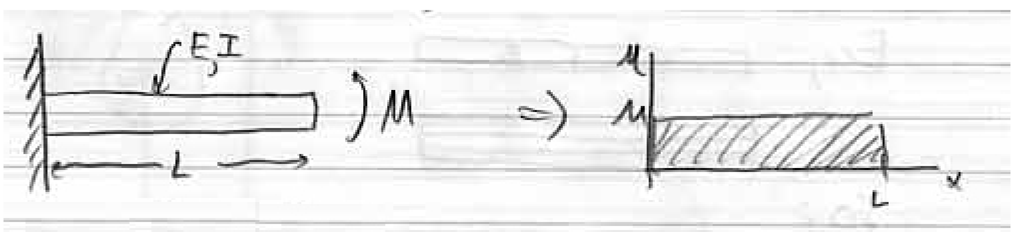
Recall:

$$\frac{1}{\rho} = \frac{M}{EI}$$

So:

$$\frac{d^2 v}{dx^2} = \frac{M(x)}{E(x)I(x)}$$

Example: Pure Bending



$$M(x) = M$$

So:

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}$$

Integrate:

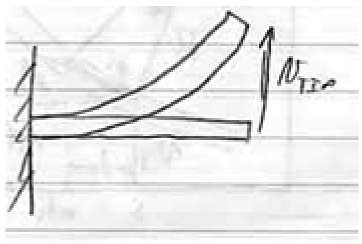
$$\frac{dv}{dx} = \frac{Mx}{EI} + c_1$$

$$v(x) = \frac{Mx^2}{2EI} + c_1 x + c_2$$

Get c_1 and c_2 from BCs.

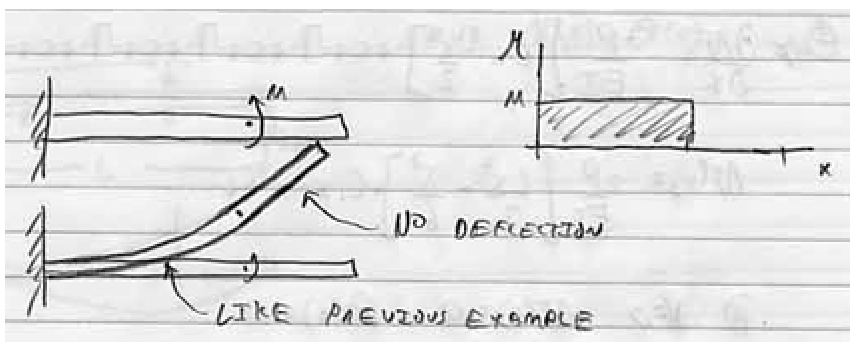
At $x = 0$, $v = 0$ due to fixed support.
At $x = 0$, $\theta = \frac{dv}{dx} = 0$ due to fixed support.
So $c_1 = c_2 = 0$
Thus:

$$v(x) = \frac{Mx^2}{2EI}$$

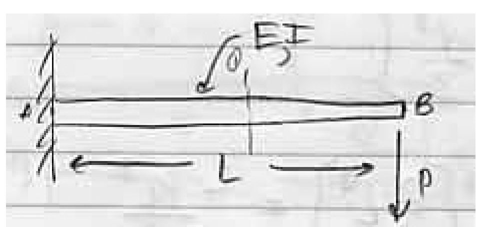


So, $v_{tip} = \frac{ML}{2EI}$, $\theta_{tip} = \frac{ML}{EI}$.

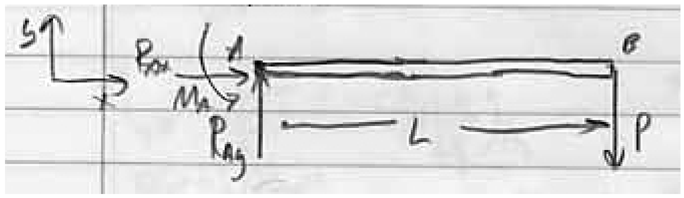
Ex (Thought)



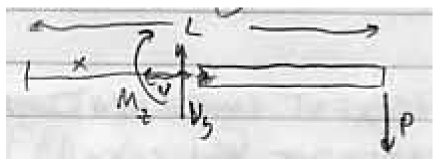
EX: End loaded cantilever beam



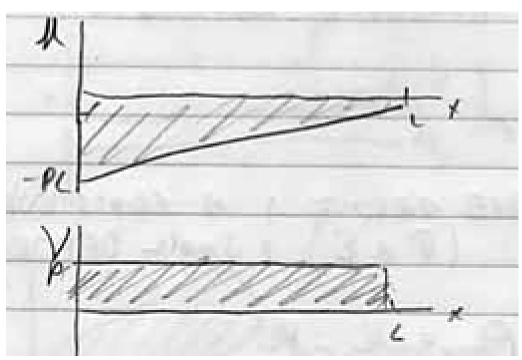
FBD



$$\sum F_x = 0 \Rightarrow R_{Ax} = 0$$
$$\sum F_y = 0 \Rightarrow R_{Ay} = P$$
$$\sum M_A = 0 \Rightarrow M_A = PL$$



$$\sum F_x = 0 \Rightarrow N = 0$$
$$\sum F_y = 0 \Rightarrow V_y = P$$
$$\sum M_* = 0 \Rightarrow M_* = -P(L - x)$$



So:

$$\frac{\partial^2 v(x)}{\partial x} = \frac{-P(L-x)}{EI}$$

$$\theta_x = \frac{\partial v}{\partial x} = \frac{-P}{EI}\left[Lx - \frac{x^2}{2}\right] + c_1$$

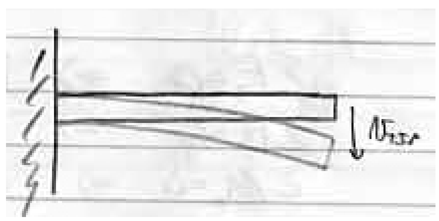
$$v(x) = \frac{-P}{EI}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + c_1 x + c_2$$

@ $x = 0$, $v(0) = 0$, $\theta(0) = 0 \Rightarrow c_1 = c_2 = 0$

So:

$$v(x) = \frac{-P}{2EI}\left[Lx^2 - \frac{x^3}{3}\right]$$

$$v_{tip} = v(L) = \frac{-P}{2EI}\left[\frac{2}{3}L^3\right] = \frac{-PL^3}{3EI}$$



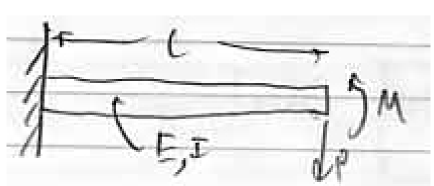
For this beam $\Rightarrow$ stiffness "F=kx"

$$k = 3\frac{EI}{L^3}$$

Different configurations have different factors.

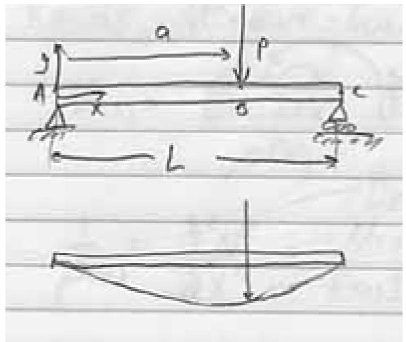
$$\theta_{tip} = \frac{-PL^2}{2EI}$$

Superpositions of these results: A consequence of linearity ($\sigma \propto \epsilon$) and small deformations

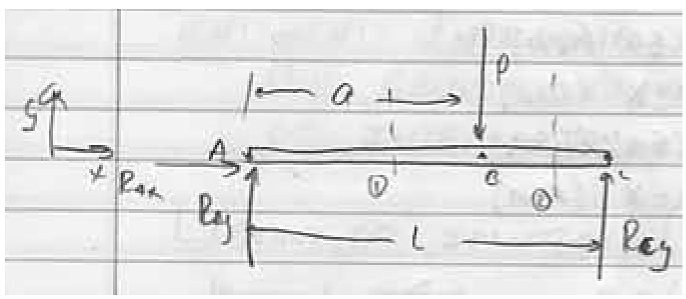


$$\theta_{tip} = \frac{ML}{EI} - \frac{PL^2}{2EI}$$

EX:



Find $v(x)$, $\theta(x)$.
FBD



$$\sum F_x = 0 \Rightarrow R_{Ax} = 0$$

$$\sum F_y = 0$$

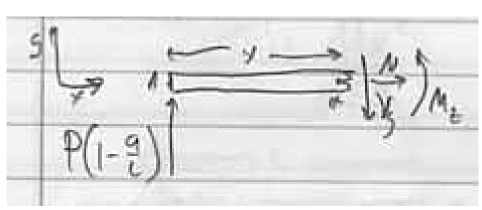
$$R_{Ay} + R_{Cy} - P = 0$$

$$\sum M_A = 0$$

$$-Pa + R_{Cy}L = 0$$

$$R_{Ay} = P(1 - \frac{a}{L})$$

FBD Cut 1 $0 \leq x \leq a$



$$\sum F_y = 0$$

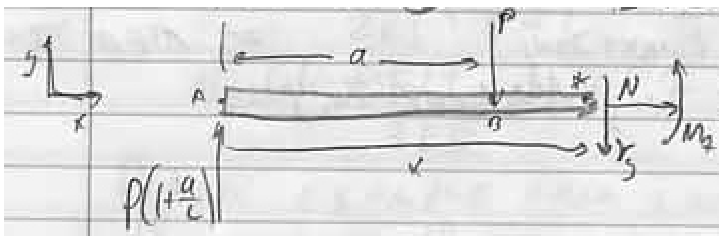
$$P(1 - \frac{a}{2}) = V_y$$

$$\sum M_* = 0$$

$$-P(1 - \frac{a}{L})x + M_z = 0$$

$$M_z = P(1 - \frac{a}{L})x$$

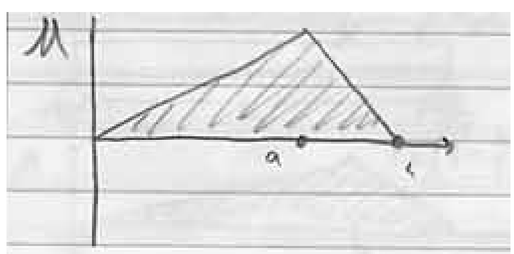
FBD Cut 2 $a \leq x \leq L$



$$\sum F_y = 0$$

$$P(1 + \frac{a}{L}) - P - V_y = 0$$

$$V_y = \frac{Pa}{L}$$



$$\sum M_* = 0$$

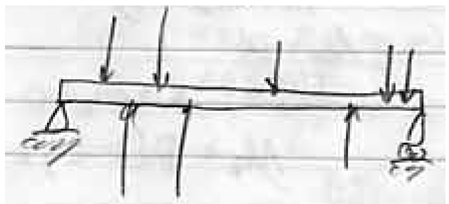
$$M_z + P(x - a) - P(1 + \frac{a}{L})x = 0$$

$$M_z = P(1 - \frac{a}{L})x - P(x - a)$$

Solution Options

1. Direct Integration
2. But what about

   Use superposition.



3. Discontinuity Functions