2.001 - MECHANICS AND MATERIALS I
Lecture #20
11/20/2006
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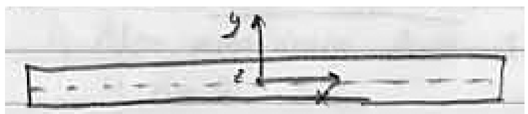
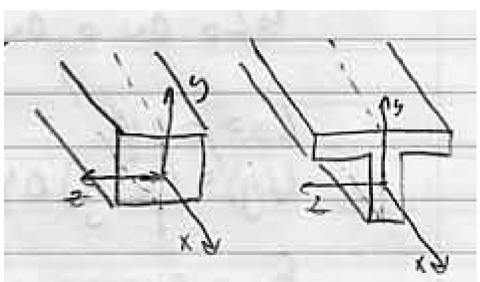
Beam Bending

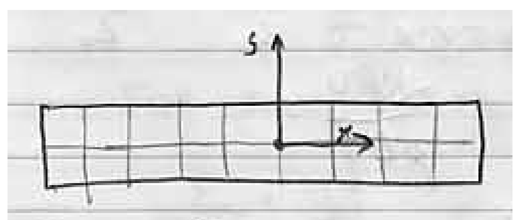
Consider a "slender" (long and thin) beam



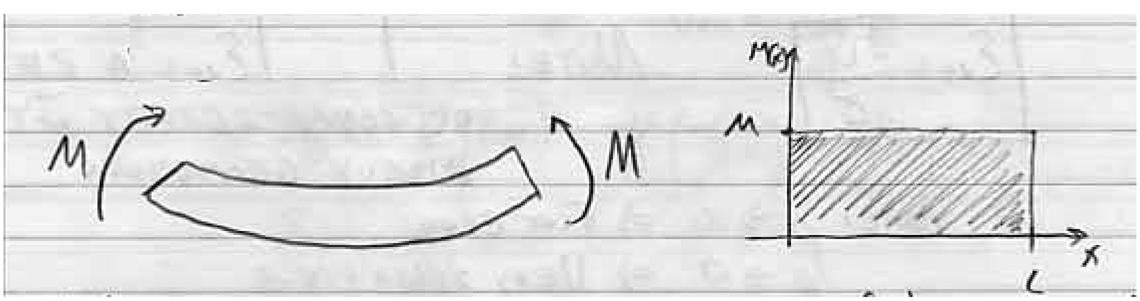
Q: What happens inside when we bend it?

Assume:
Cross-section and material properties are constant along the length
Symmetric cross-section about x-y plane

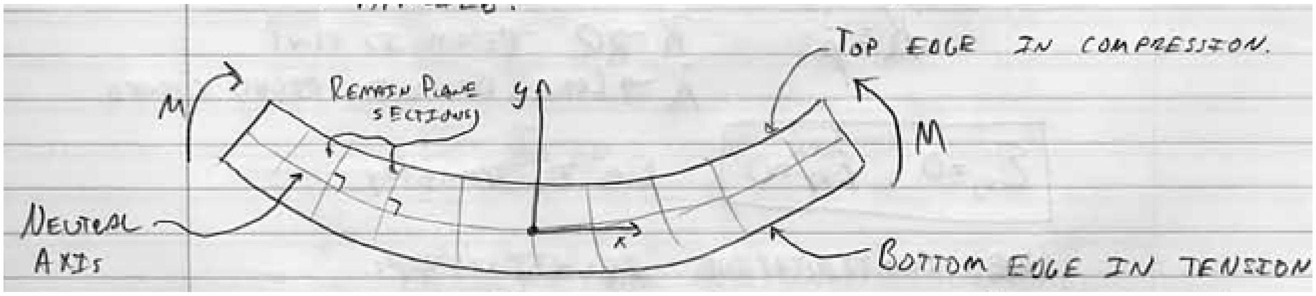




Pure Bending

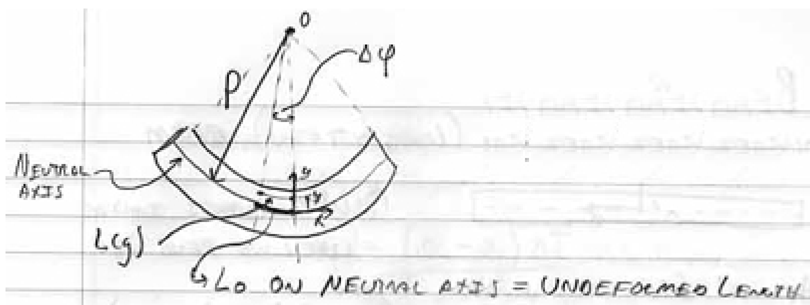


What is radius of curvature (ρ) when M is applied?



If it is in compression on one side and tension on the other, there must be a plain with no strain. This is called the neutral axis.
Note: The coordinate system is fixed such that $y = 0$ is on the neutral axis.

Compatibility



$$L_0 onneutralaxis = \text{ Undeformed Length}$$
$$L_0 = \rho\Delta\varphi$$
$$L(y) = (\rho - y)\Delta\varphi$$

$$\epsilon_{xx} = \frac{\text{Change in length in x}}{\text{Original length in x}} = \frac{\Delta L}{L_0} = \frac{L(y) - L_0}{L_0} = \frac{(\rho - y)\Delta\varphi - \rho\Delta\varphi}{\rho\Delta\varphi} = \frac{-\Delta\varphi y}{\rho\Delta\varphi}$$

$$\epsilon_{xx} = \frac{-y}{\rho}$$

Note: This is just a result of compatibility. It is purely geometric.
Note: $\rho \to \infty$: Flat Beam

$\rho \to 0$: Very Sharp Curve

Define curvature (κ)

$$\kappa = \frac{1}{\rho}$$

$\kappa \to 0$: Beam is flat
$\kappa \to \infty$: Beam is highly curved

$\epsilon_{xy} = \epsilon_{xz} = 0$ due to symmetry.

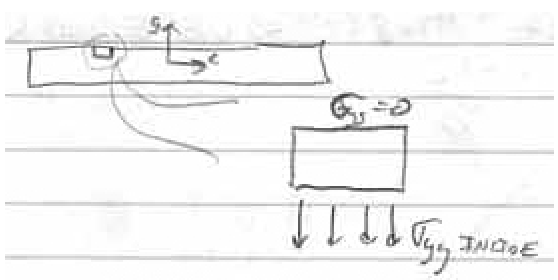
Use constitutive relationship:

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] = \frac{-y}{\rho}$$
$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] =$$
$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] =$$
$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G}$$
$$\epsilon_{yz} = \frac{\sigma_{yz}}{2G}$$
$$\epsilon_{xz} = \frac{\sigma_{xz}}{2G}$$

Known:

1. Not applying any surface tractions in y or z. So:

$$\sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0 \text{ On the surface}$$



Since the beam is thin:

$$\sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0$$

Substitute into constitutive relationships:

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} \Rightarrow \sigma_{xx} = \frac{-y}{\rho}E$$

$$\epsilon_{yy} = \frac{-\nu}{E}\epsilon_{xx} = \frac{-\nu}{E}\left(\frac{-y}{\rho}E\right) = \frac{\nu y}{\rho}$$

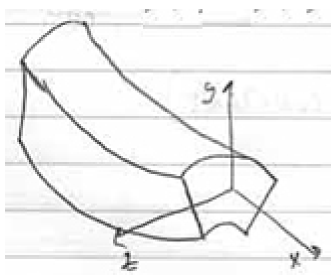
$$\epsilon_{zz} = \frac{\nu y}{\rho}$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G} = 0 \Rightarrow \sigma_{xy} = 0$$

$$\epsilon_{yz} = \frac{\sigma_{yz}}{2G} = 0 \Rightarrow \sigma_{yz} = 0$$

$$\epsilon_{xz} = \frac{\sigma_{xz}}{2G} = 0 \Rightarrow \sigma_{xz} = 0$$

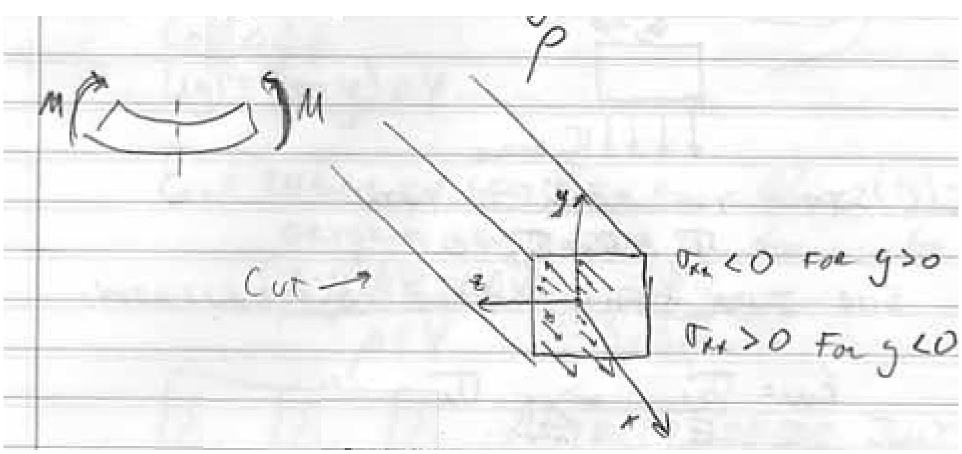
ϵ_{zz} is the anti-elastic curvature.

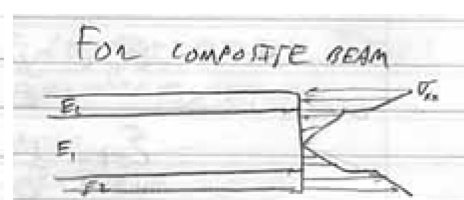


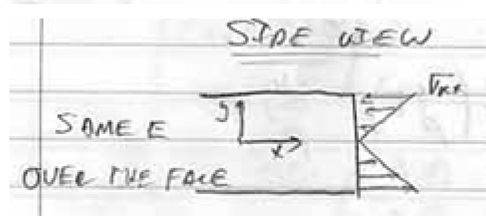
ϵ_{yy} extends the height of the compressed side and shrinks the height of the tensile side.
Where is the neutral axis? $\Rightarrow$ use equilibrium.

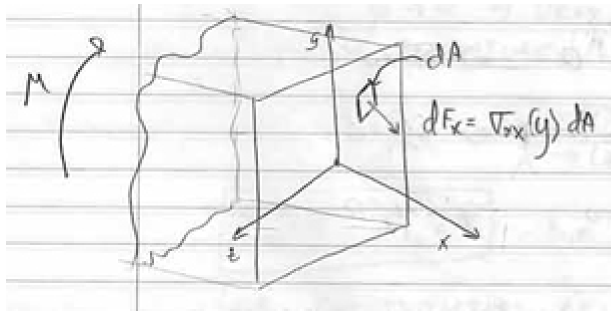
$$\epsilon_{xx} = \frac{-y}{\rho}$$

$$\sigma_{xx} = \epsilon_{xx}E = \frac{-yE}{\rho}$$









$$\sum F_x = 0$$

$$\int_A dF_x = \int_A \sigma_{xx}(y) dA = 0$$

$$\int_A \frac{-y}{\rho} E dA = 0$$

Special Case: (E is constant)

$$\frac{-E}{\rho} \int_A y dA = 0$$

So:

$$\int_A y dA = 0$$

$$\int_A y dA = \overline{y} A$$

$\overline{y}$ = distance off $y = 0$ at which the centroid lies so the neutral axis passes through the centroid of the cross-section for this special case.

$$\sum M_z = 0$$

$$-M - \int_A y df_x = 0$$

$$-M - \int_A \sigma_{xx}(y) dA = 0$$

$$-M - \int_A y(\frac{-y}{\rho} E) dA = 0$$

$$M = \int_A \frac{y^2}{\rho} E dA$$

Note: This is a general equation.
Special Case: E is contant

$$M = \frac{E}{\rho} \int_A y^2 dA$$

$$\int_A y^2 dA \text{ Moment of Inertia (I)}$$

So for special case:

$$M = EI \left(\frac{1}{\rho} \right)$$

Note similarity to "F=kx" of a spring.