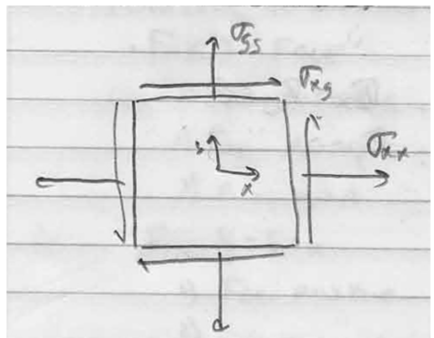
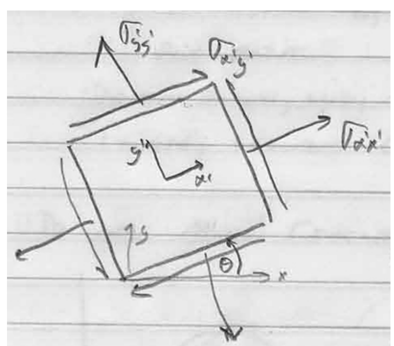
2.001 - MECHANICS AND MATERIALS I
Lecture #17
11/8/2006
Prof. Carol Livermore

Recall: Stress Transformations



$$[\sigma\ ] = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix}.$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$$



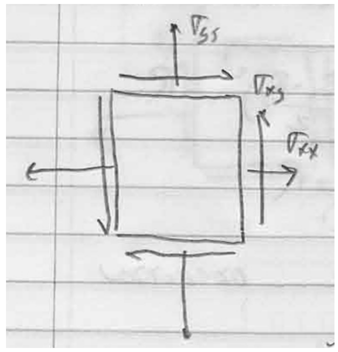
$$[\sigma\ ] = \begin{vmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{x'y'} & \sigma_{y'y'} \end{vmatrix}.$$

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} + \sigma_{yy}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \sigma_{xy} \sin 2\theta$$

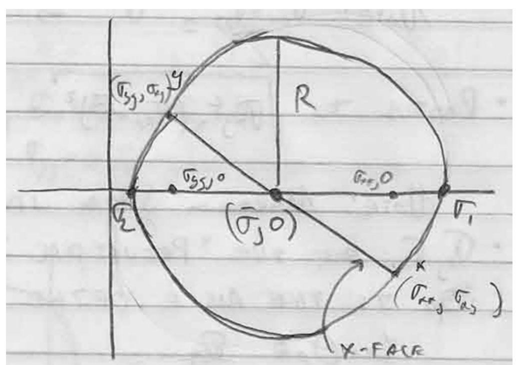
$$\sigma_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \sigma_{xy} \cos 2\theta$$

Mohr's Circle



$$\sigma_{xx} > \sigma_{yy} > 0$$

$$\sigma_{xy} > 0$$



$$C = (\overline{\sigma}, 0) = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}, 0 \right)$$

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2}$$

Principal stresses:

$$\sigma_1 = \overline{\sigma} + R$$

$$\sigma_2 = \overline{\sigma} - R$$

Mohr's circle 2θ corresponds to θ in physical space.

$$(\sigma_{x'x'} - \overline{\sigma})^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta\right)^2$$

$$\sigma_{x'y'}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta\right)^2$$

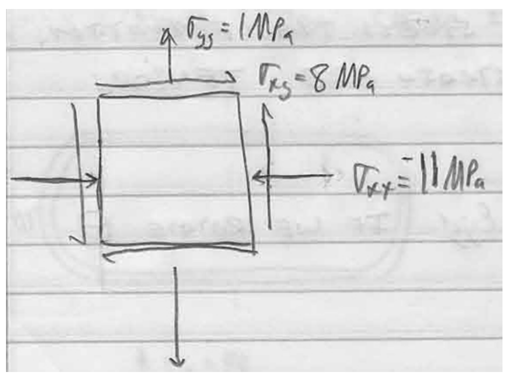
$$(\sigma_{x'y'} - \overline{\sigma})^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 \cos^2 2\theta + \sigma_{xy}^2 \sin^2 2\theta + (\sigma_{xx} - \sigma_{yy})(\sigma_{xy}\sin 2\theta)\cos 2\theta$$

$$\sigma_{x'y'}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 \sin^2 2\theta + \sigma_{xy}^2 \cos^2 2\theta - (\sigma_{xx} - \sigma_{yy})\sigma_{yy}\sin 2\theta\cos 2\theta$$

$$(\sigma_{x'x'} - \overline{\sigma})^2 + \sigma_{x'y'}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2$$

These both describe the same circle. This serves as a proof of Mohr's circles as a means to do stress transformation.

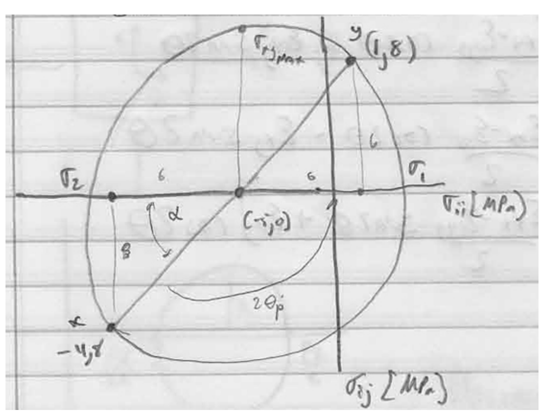
EXAMPLE:



Find:

- Max shear stress
- Direction of max shear stress
- Principal directions
- What is stress state if rotated 45°

Draw Mohr's Circle:



$$\overline{\sigma} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{-11+1}{2} = -5 \text{ MPa}$$

$$R = \sqrt{6^2 + 8^2} = 10 \text{ MPa}$$

$$\text{Max shear stress} = 10 \text{ MPa}$$

Principal Stresses

$$\sigma_1 = \overline{\sigma} + R = -5 + 10 = 5 \text{ MPa}$$

$$\sigma_2 = \overline{\sigma} + R = -5 - 10 = 15 \text{ MPa}$$

$$2\theta_p = 180 - \alpha$$

$$\tan\alpha = \frac{4}{3}$$

$$\theta_p = \frac{1}{2}(180^\circ - \tan^- 1\left(\frac{4}{3}\right))$$

Angle to max. shear:

$$\theta_{MaxShear} = \theta_p + 45^\circ$$

Rotate by 45°.

Rotate by $2 \times 45 = 90^\circ$ on Mohr's Circle and read off ansers.

$\sigma_{x'x'}$ = 3 MPa
$\sigma_{x'y'}$ = 6 MPa
$\sigma_{y'y'}$ = -13 MPa

Or use formulas:

$$\sigma_{x'x'} = -5 + \frac{(-11-1)}{2}\cos 90 + 8\sin 90 = 3$$

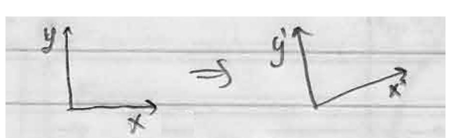
$$\sigma_{x'y'} = -\frac{(-11-1)}{2}\sin 90 + 8\cos 90 = 6$$

Strain Transformations:

This is the same as stress transformations as both stress and strain are tensors.

Given (x,y)

Ask what are $\epsilon_{x'x'}$, $\epsilon_{y'y'}$, $\epsilon_{x'y'}$ if we rotate θ to get from x-y to x'y'.



$$\epsilon_{x'x'} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} + \frac{\epsilon_{xx} - \epsilon_{yy}}{2} \cos 2\theta + \epsilon_{xy} \sin 2\theta$$

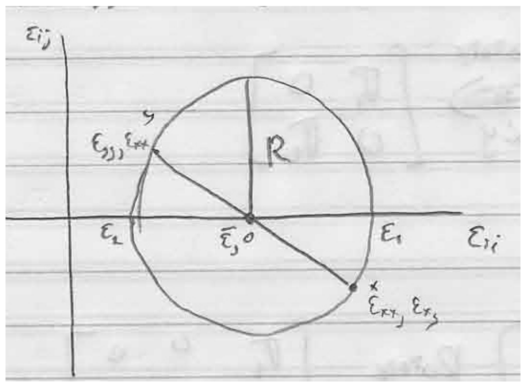
$$\epsilon_{y'y'} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} - \frac{\epsilon_{xx} - \epsilon_{yy}}{2} \cos 2\theta + \epsilon_{xy} \sin 2\theta$$

$$\epsilon_{x'y'} = -\frac{\epsilon_{xx} - \epsilon_{yy}}{2} \sin 2\theta + \epsilon_{xy} \cos 2\theta$$

Mohr's Circle for Strain

$$\epsilon_{xx} > \epsilon_{yy} > 0$$

$$\epsilon_{xy} > 0$$



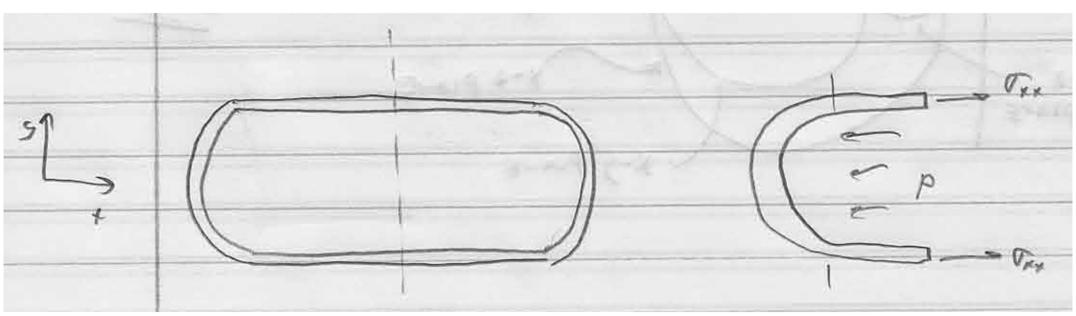
Center:

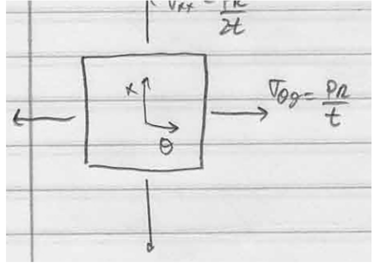
$$(\bar{\epsilon}, 0)$$

$$R = \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \epsilon_{xy}^2}$$

Note: This is entirely analagous to the Mohr's circle for stress.

EXAMPLE: Stress transformations in pressure vessels
Thin-walled cylindrical pressure vessel
Find: Max shear stress and angle

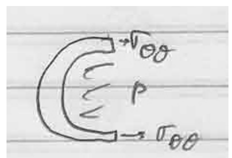




$$\sum_x F = 0$$

$$\sigma_{xx}(2\pi Rt) - p(\pi R^2) = 0$$

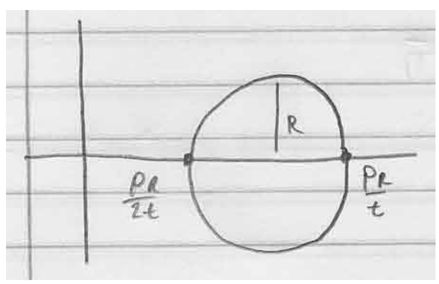
$$\sigma_{xx} = \frac{pR}{2t}$$



$$\sigma F_\theta = 0$$

$$\sigma_{\theta\theta}(L2t) - p(2RL) = 0$$

$$\sigma_{\theta\theta} = \frac{pR}{t}$$



Max Shear Stress:
Angle = 45° in physical space.

Transformation of stress and strain in 3-D
Recall for 2D

$$[\sigma\ ] = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix}.$$

Rotate to x'y'

$$[\sigma\ ] = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix}.$$

For 3-D

$$[\sigma\ ] = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

Rotate to x'y'z'

$$[\sigma\ ] = \begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix}.$$

$$\sigma_1 > \sigma_2 > \sigma_3$$