2.001 - MECHANICS AND MATERIALS I
Lecture #14
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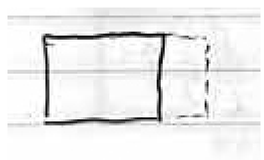
Recall from last time:
Normal strains, changes in length



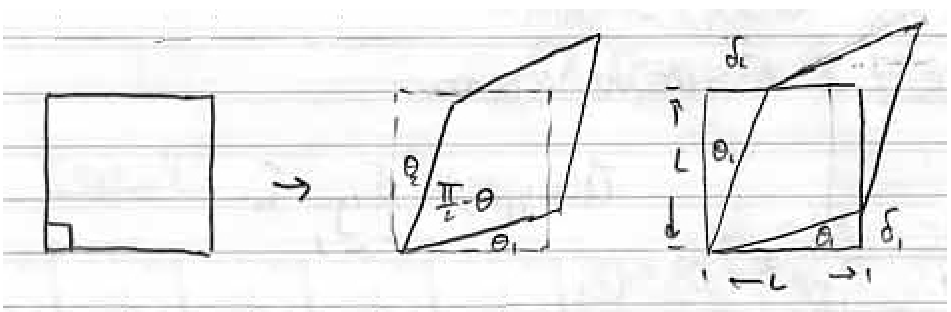
$$\vec{u}(x,y,z) = u_x(x,y,z)\hat{i} + u_y(x,y,z)\hat{j} + u_z(x,y,z)\hat{k}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}(x,y,z)$$

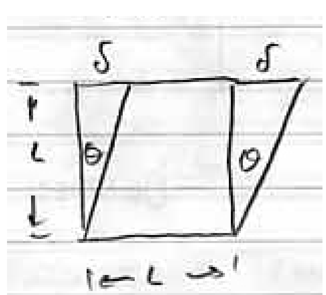
$$\epsilon_{yy} = \frac{\partial u_y}{\partial y}(x,y,z)$$

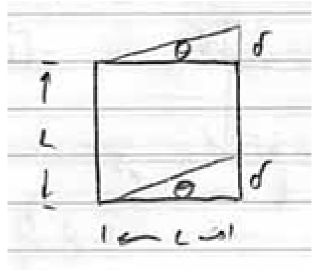
$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}(x,y,z)$$

Shear Strain



$$\gamma_{xy} = \theta = \frac{\delta}{L}$$





$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\delta_1}{L} + \frac{\delta_2}{L}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{\gamma_{xy}}{2}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

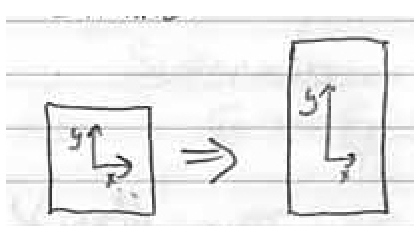
$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

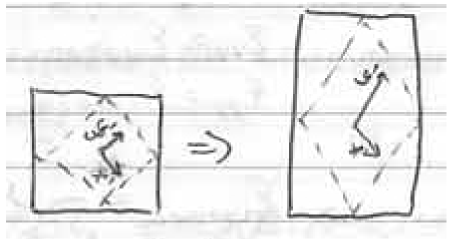
$$[\epsilon\ ] = \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{vmatrix}.$$

What kind of strain you see (normal vs. shear) and its magnitude depend on the relative orientation of deformation and coordinates.

EXAMPLE:

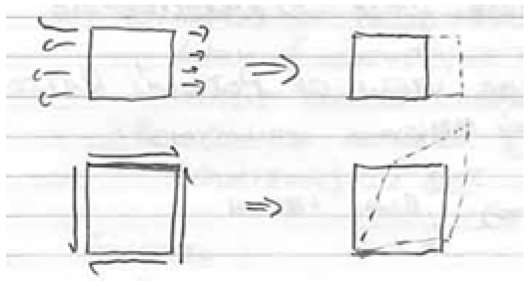


$$\epsilon_{xy} = 0$$

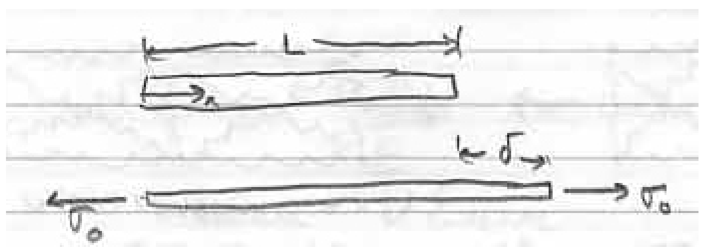


$$\epsilon_{xy} \neq 0$$

Relationship between σ and ϵ



Recall uniaxial loading:



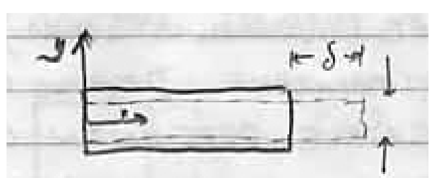
$$\epsilon = \delta/L$$

$$\epsilon_{xx} = \delta/L$$

$$\sigma = E\epsilon = \sigma_{xx} = E\epsilon_{xx}$$

$$\sigma_0 = E\epsilon_{xx}$$

Look at thicker bar:



$$\epsilon_{xx} = \frac{\delta}{L};\ \epsilon_{yy} = -(\text{something})$$

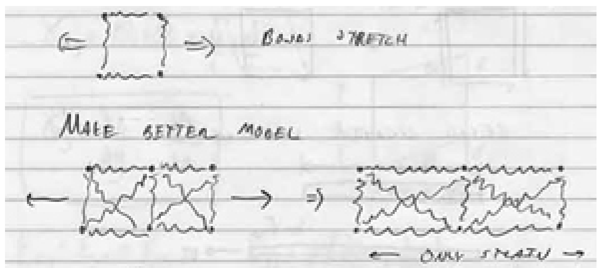
In this case:

$\epsilon_{xx} = \epsilon_{axial}$
$\epsilon_{yy} = \epsilon_{lateral}$

$$\epsilon_{lateral} = -\nu\epsilon_{axial}, \text{ where} \nu \text{ is Poisson's Ratio (unitless).}$$

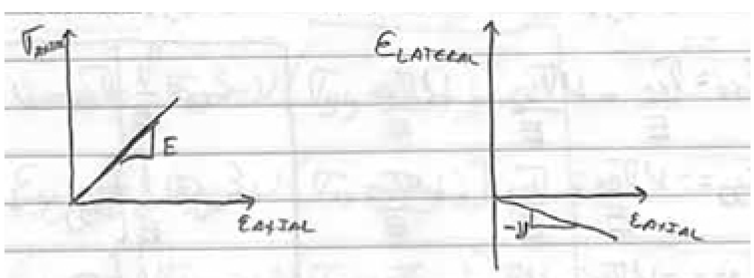
Typically $\nu \approx 0.3$
Range $0 \leq \nu \leq 0.5$
Note: $\nu = 0.5 \Rightarrow$ incompressible

Microstructure view of Poisson's Ratio
Recall Young's Modulus



Vertical bonds do not extend
Diagonal bonds do extend
May be able to minimize energy
This leads to Poisson's ratio, how this bond stretching energy is minimized.

Equations of Linear, Isotropic Elasticity
Linearity (E,ν are not a function of loading)



Superposition (A property of linearity)

$$\sigma_a \Rightarrow \epsilon_a$$

$$\sigma_b \Rightarrow \epsilon_b$$

$$\alpha\sigma_a + \beta\sigma_b \Rightarrow \alpha\epsilon_a + \beta\epsilon_b \ \alpha \text{ and } \beta \text{ are scalar constants.}$$

Isotropic:

Material properties are the same in all orientations.

Examples of anisotropic materials

Wood (against the grain, with the grain)

Single crystals (depends on which crystal direction)

Relationship between σ and ϵ (the constitutive equations) is *not* orientation dependent.

Elastic:

Deformation is removed when load is released (deformation is fully recoverable)

$$\epsilon_{axial} = \frac{\sigma_{axial}}{E} \text{ in uniaxial loading}$$
$$\epsilon_{lateral} = -\nu\epsilon_{axial}$$

Consider:

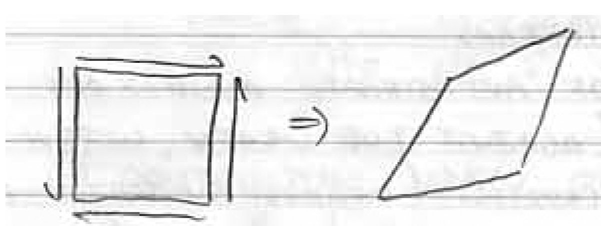
$$\sigma_{xx}, \sigma_{yy}, \sigma_{zz} \neq -$$
$$\sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0$$

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}$$
$$\epsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}$$
$$\epsilon_{zz} = -\nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E}$$

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$$
$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right]$$
$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]$$

Multi-axial stress-strain relationships among normal stresses and strains for linear isotropic elastic materials.

What about shear stresses and strains?



$$\sigma_{xy} = G\gamma_{xy} \text{ where } G \text{ is the shear modulus with units of Pa}$$

$$G = \frac{E}{2(1+\nu)}$$

For linear, isotropic elastic materials, 2 material constants fully define a material.

$$\gamma_{xy} = \frac{1}{G}\sigma_{xy}$$

$$\gamma_{xz} = \frac{1}{G}\sigma_{xz}$$

$$\gamma_{yz} = \frac{1}{G}\sigma_{yz}$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$

$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}$$

$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

Equations of linear isotropic elasticity (aka. Constitutive Relationships)

$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\Big]$$

$$\epsilon_{yy} = \frac{1}{E}\Big[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\Big]$$

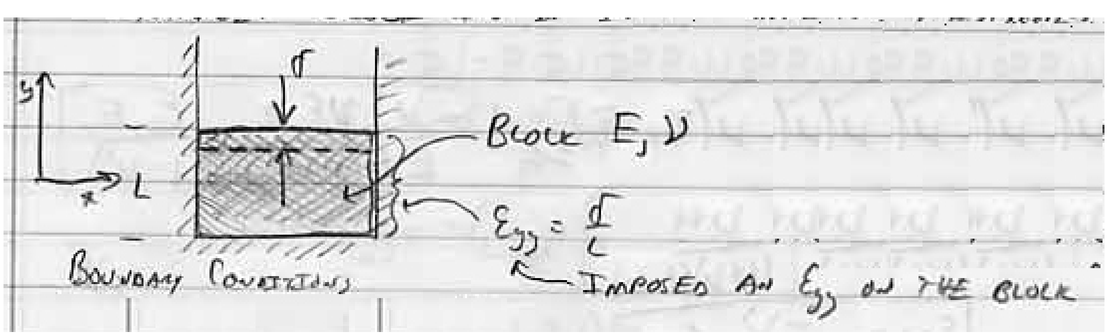
$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\Big]$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$

$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}$$

$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

EXAMPLE: Block in a frictionless channel



What are all stresses and strains?
No shears due to frictionless

Boundary Conditions:

$$\begin{array}{ll} \epsilon_{xx} = 0 & \sigma_{xx} = ? \\ \epsilon_{yy} = \text{Given} & \sigma_{yy} = ? \\ \epsilon_{zz} = ? & \sigma_{zz} = 0 \end{array}$$

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$$

So:

$$0 = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$$

$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right]$$

So:

$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]$$

So:

$$\epsilon_{zz} = \frac{1}{E}\left[-\nu(\sigma_{xx} + \sigma_{yy})\right]$$

Solve:

$$\sigma_{xx} = \nu\sigma_{yy}$$

Plug in:

$$\begin{aligned} \epsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\nu\sigma_{yy})\right] \\ &= \frac{1}{E}\left[\sigma_{yy} - \nu^2\sigma_{yy}\right] \end{aligned}$$

$$\sigma_{yy} = \frac{\epsilon_{yy}E}{(1-\nu^2)}$$

Plug in:

$$\sigma_{xx} = \frac{\nu \epsilon_{yy} E}{(1-\nu^2)}$$

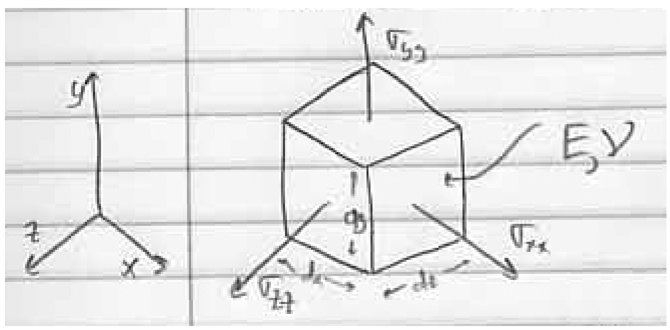
Plug in:

$$\epsilon_{zz} = \frac{1}{E}\left[-\nu(\sigma_{xx}+\sigma_{yy}) - \frac{\nu}{E}\left[\frac{\nu \epsilon_{yy} E}{(1-\nu^2)} + \frac{\epsilon_{yy} E}{(1-\nu^2)}\right]\right]$$

$$\epsilon_{zz} = -\nu\left[\frac{\nu+1}{(1+\nu)(1-\nu)}\right]$$

$$\epsilon_{zz} = \frac{-\nu}{1-\nu}\epsilon_{yy}$$

EXAMPLE: Hydrostatic Pressure



Q: What is change in volume?

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$$

Initial Volume:

$$V_i = dxdydz$$

Final Volume:

$$(1+\epsilon_{xx})dx(1+\epsilon_{yy})dy(1+\epsilon_{zz})dz$$

$$\Delta V = V_f - V_i$$

$$\Delta V = (1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{zz})V_i - V_i$$

For small strains:

$$\epsilon^2, \epsilon^3 \approx 0$$

So:

$$(1+\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})V_i - V_i$$

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\right]$$

$$= \frac{1}{E}\left[-p - \nu(-p-p)\right]$$

$$\epsilon_{xx} = \frac{-1(1-2\nu)}{E}p$$

$$\epsilon_{yy} = \frac{-1(1-2\nu)}{E}p$$

$$\epsilon_{zz} = \frac{-1(1-2\nu)}{E}p$$

$$\Delta V\left[1 - \frac{3(1-2\nu)}{E}p\right]V_i - V_i = \frac{-3(1-2\nu)}{E}pV_i$$

$$\frac{\Delta V}{V_i} = \frac{-3(1-2\nu)}{E}p$$

$$\frac{-p}{\Delta V/V_i} = \frac{E}{3(1-2\nu)} = k$$

k is the bulk modulus.

Note: If $\nu = 0.5$, $\frac{\Delta V}{V_i} = 0$ (Incompressible) and $k \to \infty$