18.S997 Spring 2015: Problem Set 2

Problem 2.1

Let $X = (1, Z, \ldots, Z^{d-1})^\top \in \mathbb{R}^d$ be a random vector where Z is a random variable. Show that the matrix $\mathbb{E}(XX^\top)$ is positive definite if Z admits a probability density with respect to the Lebesgue measure on $\mathbb{R}$.

Problem 2.2

For any $q > 0$, a vector $\theta \in \mathbb{R}^d$ is said to be in a weak ℓ_q ball of radius R if the decreasing rearrangement $|\theta_{[1]}| \ge |\theta_{[2]}| \ge \ldots$ satisfies

$$|\theta_{[j]}| \le R j^{-1/q} .$$

Moreover, we define the weak ℓ_q norm of θ by

$$|\theta|_{w\ell_q} = \max_{1 \le j \le d} j^{1/q} |\theta_{[j]}|$$

(a) Give examples of $\theta, \theta' \in \mathbb{R}^d$ such that

$$|\theta + \theta'|_{w\ell_1} > |\theta|_{w\ell_1} + |\theta'|_{w\ell_1}$$

What do you conclude?

(b) Show that $|\theta|_{w\ell_q} \le |\theta|_q$.

(c) Show that if $\lim_{d\to\infty} |\theta|_{w\ell_q} < \infty$, then $\lim_{d\to\infty} |\theta|_{q'} < \infty$ for all $q' > q$.

(d) Show that, for any $q \in (0, 2)$ if $\lim_{d\to\infty} |\theta|_{w\ell_q} = C$, there exists a constant $C_q > 0$ that depends on q but not on d and such that under the assumptions of Theorem 2.11, it holds

$$|\hat\theta^{\text{HRD}} - \theta^*|_2^2 \le C_q \Big(\frac{\sigma^2 \log 2d}{n}\Big)^{1-\frac{q}{2}}$$

with probability .99.

Problem 2.3

Assume that the linear model (Equation 2.2) with $\varepsilon \sim \mathsf{subG}_n(\sigma^2)$ and $\theta^* \neq 0$. Show that the modified BIC estimator $\hat{\theta}$ defined by

$$\hat{\theta} \in \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \left\{ \frac{1}{n}|Y - \mathbb{X}\theta|_2^2 + \lambda |\theta|_0 \log\left(\frac{ed}{|\theta|_0}\right) \right\}$$

satisfies

$$\mathsf{MSE}(\mathbb{X}\hat{\theta}) \lesssim |\theta^*|_0 \sigma^2 \frac{\log\left(\frac{ed}{|\theta^*|_0}\right)}{n}.$$

with probability .99, for appropriately chosen λ. What do you conclude?

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