

Chapter 2

The category of sets

The theory of sets was invented as a foundation for all of mathematics. The notion of sets and functions serves as a basis on which to build our intuition about categories in general. In this chapter we will give examples of sets and functions and then move on to discuss commutative diagrams. At this point we can introduce ologs which will allow us to use the language of category theory to speak about real world concepts. Then we will introduce limits and colimits, and their universal properties. All of this material is basic set theory, but it can also be taken as an investigation of our first category, the *category of sets*, which we call **Set**. We will end this chapter with some other interesting constructions in **Set** that do not fit into the previous sections.

2.1 Sets and functions

2.1.1 Sets

In this course I'll assume you know what a set is. We can think of a set X as a collection of things $x \in X$, each of which is recognizable as being in X and such that for each pair of named elements $x, x' \in X$ we can tell if $x = x'$ or not.¹ The set of pendulums is the collection of things we agree to call pendulums, each of which is recognizable as being a pendulum, and for any two people pointing at pendulums we can tell if they're pointing at the same pendulum or not.

Notation 2.1.1.1. The symbol \emptyset denotes the set with no elements. The symbol \mathbb{N} denotes the set of natural numbers, which we can write as

$$\mathbb{N} := \{0, 1, 2, 3, 4, \dots, 877, \dots\}.$$

The symbol \mathbb{Z} denotes the set of integers, which contains both the natural numbers and their negatives,

$$\mathbb{Z} := \{\dots, -551, \dots, -2, -1, 0, 1, 2, \dots\}.$$

If A and B are sets, we say that A is a *subset* of B , and write $A \subseteq B$, if every element of A is an element of B . So we have $\mathbb{N} \subseteq \mathbb{Z}$. Checking the definition, one sees that

¹Note that the symbol x' , read “x-prime”, has nothing to do with calculus or derivatives. It is simply notation that we use to name a symbol that is suggested as being somehow like x . This suggestion of kinship between x and x' is meant only as an aid for human cognition, and not as part of the mathematics.

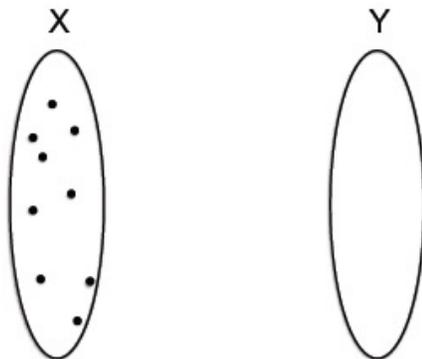


Figure 2.1: A set X with 9 elements and a set Y with no elements, $Y = \emptyset$.

for any set A , we have (perhaps uninteresting) subsets $\emptyset \subseteq A$ and $A \subseteq A$. We can use *set-builder notation* to denote subsets. For example the set of even integers can be written $\{n \in \mathbb{Z} \mid n \text{ is even}\}$. The set of integers greater than 2 can be written in many ways, such as

$$\{n \in \mathbb{Z} \mid n > 2\} \quad \text{or} \quad \{n \in \mathbb{N} \mid n > 2\} \quad \text{or} \quad \{n \in \mathbb{N} \mid n \geq 3\}.$$

The symbol \exists means “there exists”. So we could write the set of even integers as

$$\{n \in \mathbb{Z} \mid n \text{ is even}\} \quad = \quad \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z} \text{ such that } 2m = n\}.$$

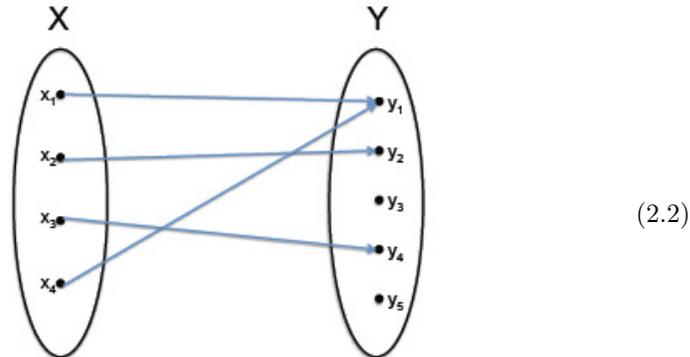
The symbol $\exists!$ means “there exists a unique”. So the statement “ $\exists! x \in \mathbb{R}$ such that $x^2 = 0$ ” means that there is one and only one number whose square is 0. Finally, the symbol \forall means “for all”. So the statement “ $\forall m \in \mathbb{N} \exists n \in \mathbb{N}$ such that $m < n$ ” means that for every number there is a bigger one.

As you may have noticed, we use the colon-equals notation “ $A := XYZ$ ” to mean something like “define A to be XYZ ”. That is, a colon-equals declaration is not denoting a fact of nature (like $2 + 2 = 4$), but a choice of the speaker. It just so happens that the notation above, such as $\mathbb{N} := \{0, 1, 2, \dots\}$, is a widely-held choice.

Exercise 2.1.1.2. Let $A = \{1, 2, 3\}$. What are all the subsets of A ? Hint: there are 8. \diamond

2.1.2 Functions

If X and Y are sets, then a *function f from X to Y* , denoted $f: X \rightarrow Y$, is a mapping that sends each element $x \in X$ to an element of Y , denoted $f(x) \in Y$. We call X the *domain* of the function f and we call Y the *codomain* of f .



Note that for every element $x \in X$, there is exactly one arrow emanating from x , but for an element $y \in Y$, there can be several arrows pointing to y , or there can be no arrows pointing to y .

Application 2.1.2.1. In studying the mechanics of materials, one wishes to know how a material responds to tension. For example a rubber band responds to tension differently than a spring does. To each material we can associate a [force-extension curve](#), recording how much force the material carries when extended to various lengths. Once we fix a methodology for performing experiments, finding a material's force-extension curve would ideally constitute a function from the set of materials to the set of curves. ²

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Exercise 2.1.2.2. Here is a simplified account of how the [brain receives light](#). The eye contains about 100 million photoreceptor (PR) cells. Each connects to a retinal ganglion (RG) cell. No PR cell connects to two different RG cells, but usually many PR cells can attach to a single RG cell.

Let PR denote the set of photoreceptor cells and let RG denote the set of retinal ganglion cells.

- a.) According to the above account, does the connection pattern constitute a function $RG \rightarrow PR$, a function $PR \rightarrow RG$ or neither one?
- b.) Would you guess that the connection pattern that exists between other areas of the brain are "function-like"?

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Example 2.1.2.3. Suppose that X is a set and $X' \subseteq X$ is a subset. Then we can consider the function $X' \rightarrow X$ given by sending every element of X' to "itself" as an element of X . For example if $X = \{a, b, c, d, e, f\}$ and $X' = \{b, d, e\}$ then $X' \subseteq X$ and we turn that into the function $X' \rightarrow X$ given by $b \mapsto b, d \mapsto d, e \mapsto e$. ³

As a matter of notation, we may sometimes say something like the following: Let X be a set and let $i: X' \subseteq X$ be a subset. Here we are making clear that X' is a subset of X , but that i is the name of the associated function.

²In reality, different samples of the same material, say samples of different sizes or at different temperatures, may have different force-extension curves. If we want to see this as a true function whose codomain is curves it should have as domain something like the set of material samples.

³This kind of arrow, \mapsto , is read aloud as "maps to". A function $f: X \rightarrow Y$ means a rule for assigning to each element $x \in X$ an element $f(x) \in Y$. We say that " x maps to $f(x)$ " and write $x \mapsto f(x)$.

Exercise 2.1.2.4. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be the function that sends every natural number to its square, e.g. $f(6) = 36$. First fill in the blanks below, then answer a question.

- $2 \mapsto$ _____
- $0 \mapsto$ _____
- $-2 \mapsto$ _____
- $5 \mapsto$ _____
- Consider the symbol \rightarrow and the symbol \mapsto . What is the difference between how these two symbols are used in this book?

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Given a function $f: X \rightarrow Y$, the elements of Y that have at least one arrow pointing to them are said to be *in the image* of f ; that is we have

$$\text{im}(f) := \{y \in Y \mid \exists x \in X \text{ such that } f(x) = y\}. \quad (2.3)$$

Exercise 2.1.2.5. If $f: X \rightarrow Y$ is depicted by (2.2) above, write its image, $\text{im}(f)$ as a set.

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Given a function $f: X \rightarrow Y$ and a function $g: Y \rightarrow Z$, where the codomain of f is the same set as the domain of g (namely Y), we say that f and g are composable

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

The *composition* of f and g is denoted by $g \circ f: X \rightarrow Z$.

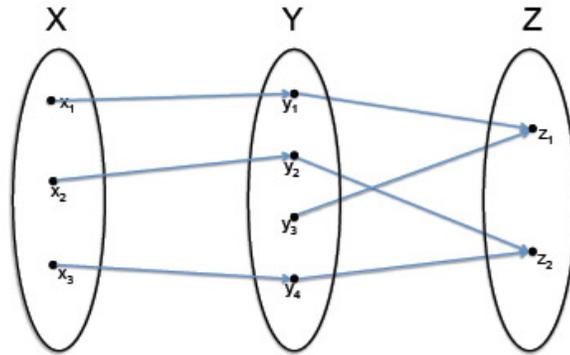


Figure 2.4: Functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ compose to a function $g \circ f: X \rightarrow Z$; just follow the arrows.

Let X and Y be sets. We write $\text{Hom}_{\text{Set}}(X, Y)$ to denote the set of functions $X \rightarrow Y$.
⁴ Note that two functions $f, g: X \rightarrow Y$ are equal if and only if for every element $x \in X$ we have $f(x) = g(x)$.

Exercise 2.1.2.6. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y\}$.

⁴The strange notation $\text{Hom}_{\text{Set}}(-, -)$ will make more sense later, when it is seen as part of a bigger story.

- a.) How many elements does $\text{Hom}_{\mathbf{Set}}(A, B)$ have?
 b.) How many elements does $\text{Hom}_{\mathbf{Set}}(B, A)$ have?

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Exercise 2.1.2.7.

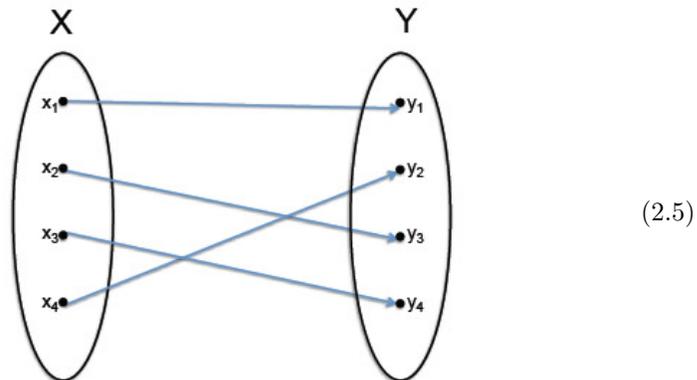
- a.) Find a set A such that for all sets X there is exactly one element in $\text{Hom}_{\mathbf{Set}}(X, A)$.
 Hint: draw a picture of proposed A 's and X 's.
 b.) Find a set B such that for all sets X there is exactly one element in $\text{Hom}_{\mathbf{Set}}(B, X)$.

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For any set X , we define the *identity function on X* , denoted $\text{id}_X: X \rightarrow X$, to be the function such that for all $x \in X$ we have $\text{id}_X(x) = x$.

Definition 2.1.2.8 (Isomorphism). Let X and Y be sets. A function $f: X \rightarrow Y$ is called an *isomorphism*, denoted $f: X \xrightarrow{\cong} Y$, if there exists a function $g: Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. We also say that f is *invertible* and we say that g is *the inverse* of f . If there exists an isomorphism $X \xrightarrow{\cong} Y$ we say that X and Y are *isomorphic* sets and may write $X \cong Y$.

Example 2.1.2.9. If X and Y are sets and $f: X \rightarrow Y$ is an isomorphism then the analogue of Diagram 2.2 will look like a perfect matching, more often called a *one-to-one correspondence*. That means that no two arrows will hit the same element of Y , and every element of Y will be in the image. For example, the following depicts an isomorphism $X \xrightarrow{\cong} Y$.



Application 2.1.2.10. There is an isomorphism between the set Nuc_{DNA} of **nucleotides** found in DNA and the set Nuc_{RNA} of nucleotides found in RNA. Indeed both sets have four elements, so there are 24 different isomorphisms. But only one is useful. Before we say which one it is, let us say there is also an isomorphism $\text{Nuc}_{\text{DNA}} \cong \{A, C, G, T\}$ and an isomorphism $\text{Nuc}_{\text{RNA}} \cong \{A, C, G, U\}$, and we will use the letters as abbreviations for the nucleotides.

The convenient isomorphism $\text{Nuc}_{\text{DNA}} \xrightarrow{\cong} \text{Nuc}_{\text{RNA}}$ is that given by RNA transcription; it sends

$$A \mapsto U, C \mapsto G, G \mapsto C, T \mapsto A.$$

(See also Application 4.1.2.19.) There is also an isomorphism $\text{Nuc}_{\text{DNA}} \xrightarrow{\cong} \text{Nuc}_{\text{DNA}}$ (the matching in the double-helix) given by

$$A \mapsto T, C \mapsto G, G \mapsto C, T \mapsto A.$$

Protein production can be modeled as a function from the set of 3-nucleotide sequences to the set of eukaryotic amino acids. However, it cannot be an isomorphism because there are $4^3 = 64$ triplets of RNA nucleotides, but only 21 eukaryotic amino acids.

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Exercise 2.1.2.11. Let $n \in \mathbb{N}$ be a natural number and let X be a set with exactly n elements.

- a.) How many isomorphisms are there from X to itself?
- b.) Does your formula from part a.) hold when $n = 0$?

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Lemma 2.1.2.12. *The following facts hold about isomorphism.*

1. Any set A is isomorphic to itself; i.e. there exists an isomorphism $A \xrightarrow{\cong} A$.
2. For any sets A and B , if A is isomorphic to B then B is isomorphic to A .
3. For any sets A, B , and C , if A is isomorphic to B and B is isomorphic to C then A is isomorphic to C .

Proof. 1. The identity function $\text{id}_A: A \rightarrow A$ is invertible; its inverse is id_A because $\text{id}_A \circ \text{id}_A = \text{id}_A$.

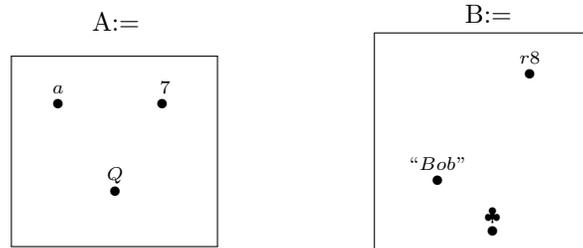
2. If $f: A \rightarrow B$ is invertible with inverse $g: B \rightarrow A$ then g is an isomorphism with inverse f .

3. If $f: A \rightarrow B$ and $f': B \rightarrow C$ are each invertible with inverses $g: B \rightarrow A$ and $g': C \rightarrow B$ then the following calculations show that $f' \circ f$ is invertible with inverse $g \circ g'$:

$$\begin{aligned} (f' \circ f) \circ (g \circ g') &= f' \circ (f \circ g) \circ g' = f' \circ \text{id}_B \circ g' = f' \circ g' = \text{id}_C \\ (g \circ g') \circ (f' \circ f) &= g \circ (g' \circ f') \circ f = g \circ \text{id}_B \circ f = g \circ f = \text{id}_A \end{aligned}$$

□

Exercise 2.1.2.13. Let A and B be the sets drawn below:



Note that the sets A and B are isomorphic. Supposing that $f: B \rightarrow \{1, 2, 3, 4, 5\}$ sends “Bob” to 1, sends ♣ to 3, and sends $r8$ to 4, is there a canonical function $A \rightarrow \{1, 2, 3, 4, 5\}$ corresponding to f ?⁵ \diamond

Exercise 2.1.2.14. Find a set A such that for any set X there is an isomorphism of sets

$$X \cong \text{Hom}_{\text{Set}}(A, X).$$

Hint: draw a picture of proposed A 's and X 's. \diamond

For any natural number $n \in \mathbb{N}$, define a set

$$\underline{n} := \{1, 2, 3, \dots, n\}. \quad (2.6)$$

So, in particular, $\underline{2} = \{1, 2\}$, $\underline{1} = \{1\}$, and $\underline{0} = \emptyset$.

Let A be any set. A function $f: \underline{n} \rightarrow A$ can be written as a sequence

$$f = (f(1), f(2), \dots, f(n)).$$

Exercise 2.1.2.15.

a.) Let $A = \{a, b, c, d\}$. If $f: \underline{10} \rightarrow A$ is given by $(a, b, c, c, b, a, d, d, a, b)$, what is $f(4)$?

b.) Let $s: \underline{7} \rightarrow \mathbb{N}$ be given by $s(i) = i^2$. Write s out as a sequence. \diamond

Definition 2.1.2.16. Cardinality of finite sets][

Let A be a set and $n \in \mathbb{N}$ a natural number. We say that A is *has cardinality* n , denoted

$$|A| = n,$$

if there exists an isomorphism of sets $A \cong \underline{n}$. If there exists some $n \in \mathbb{N}$ such that A has cardinality n then we say that A is *finite*. Otherwise, we say that A is *infinite* and write $|A| \geq \infty$.

Exercise 2.1.2.17.

a.) Let $A = \{5, 6, 7\}$. What is $|A|$?

b.) What is $|\mathbb{N}|$?

c.) What is $|\{n \in \mathbb{N} \mid n \leq 5\}|$? \diamond

Lemma 2.1.2.18. *Let A and B be finite sets. If there is an isomorphism of sets $f: A \rightarrow B$ then the two sets have the same cardinality, $|A| = |B|$.*

Proof. Suppose $f: A \rightarrow B$ is an isomorphism. If there exists natural numbers $m, n \in \mathbb{N}$ and isomorphisms $a: \underline{m} \xrightarrow{\cong} A$ and $b: \underline{n} \xrightarrow{\cong} B$ then $\underline{m} \xrightarrow{a^{-1}} A \xrightarrow{f} B \xrightarrow{b} \underline{n}$ is an isomorphism. One can prove by induction that the sets \underline{m} and \underline{n} are isomorphic if and only if $m = n$. \square

⁵Canonical means something like “best choice”, a choice that stands out as the only reasonable one.

2.2 Commutative diagrams

At this point it is difficult to precisely define diagrams or commutative diagrams in general, but we can give the heuristic idea. ⁶ Consider the following picture:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow h & \downarrow g \\
 & & C
 \end{array}
 \tag{2.7}$$

We say this is a *diagram of sets* if each of A, B, C is a set and each of f, g, h is a function. We say this diagram *commutes* if $g \circ f = h$. In this case we refer to it as a commutative triangle of sets.

Application 2.2.1.1. The **central dogma of molecular biology** is that “DNA codes for RNA codes for protein”. That is, there is a function from DNA triplets to RNA triplets and a function from RNA triplets to amino acids. But sometimes we just want to discuss the translation from DNA to amino acids, and this is the composite of the other two. The commutative diagram is a picture of this fact.

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Consider the following picture:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h \downarrow & & \downarrow g \\
 C & \xrightarrow{i} & D
 \end{array}$$

We say this is a *diagram of sets* if each of A, B, C, D is a set and each of f, g, h, i is a function. We say this diagram *commutes* if $g \circ f = i \circ h$. In this case we refer to it as a commutative square of sets.

Application 2.2.1.2. Given a physical system S , there may be two mathematical approaches $f: S \rightarrow A$ and $g: S \rightarrow B$ that can be applied to it. Either of those results in a prediction of the same sort, $f': A \rightarrow P$ and $g': B \rightarrow P$. For example, in **mechanics** we can use either Lagrangian approach or the Hamiltonian approach to predict future states. To say that the diagram

$$\begin{array}{ccc}
 S & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 B & \longrightarrow & P
 \end{array}$$

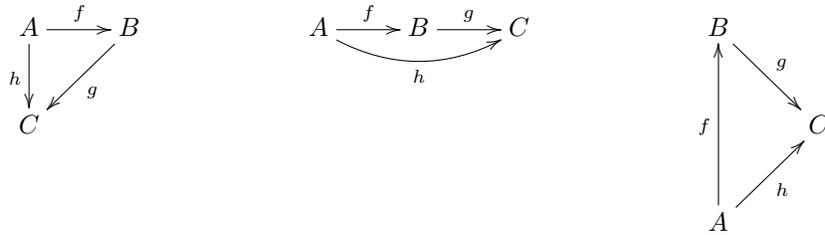
commutes would say that these approaches give the same result.

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And so on. Note that diagram (2.7) is considered to be the same diagram as each of

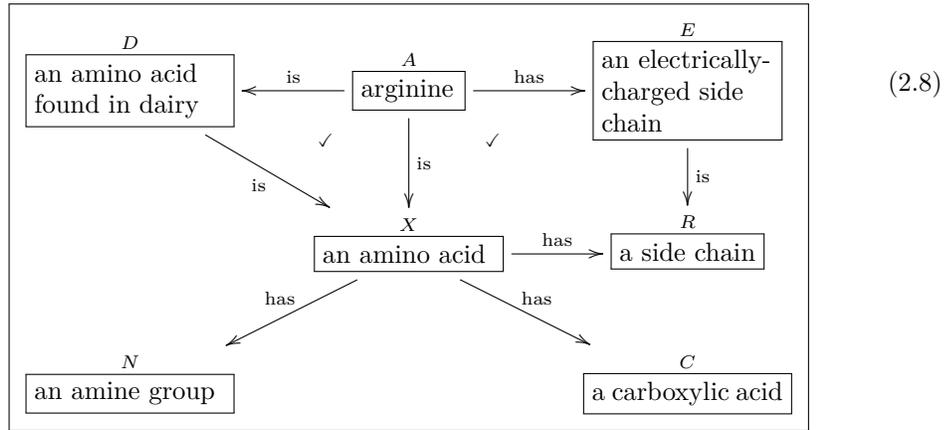
⁶We will define commutative diagrams precisely in Section 4.5.2.

the following:



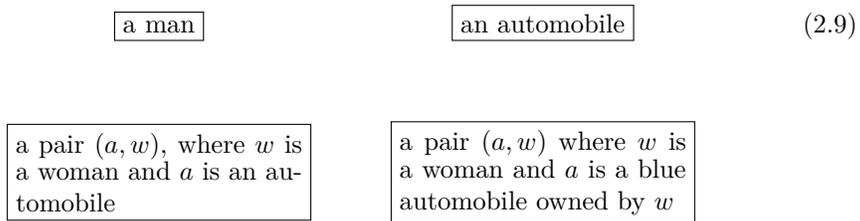
2.3 Ologs

In this course we will ground the mathematical ideas in applications whenever possible. To that end we introduce ologs, which will serve as a bridge between mathematics and various conceptual landscapes. The following material is taken from [SK], an introduction to ologs.



2.3.1 Types

A type is an abstract concept, a distinction the author has made. We represent each type as a box containing a *singular indefinite noun phrase*. Each of the following four boxes is a type:



Each of the four boxes in (2.9) represents a type of thing, a whole class of things, and the label on that box is what one should call *each example* of that class. Thus “a man” does not represent a single man, but the set of men, each example of which is called “a man”. Similarly, the bottom right box represents an abstract type of thing,

which probably has more than a million examples, but the label on the box indicates the common name for each such example.

Typographical problems emerge when writing a text-box in a line of text, e.g. the text-box a man seems out of place here, and the more in-line text-boxes there are, the worse it gets. To remedy this, I will denote types which occur in a line of text with corner-symbols; e.g. I will write \ulcorner “a man” \urcorner instead of a man.

2.3.1.1 Types with compound structures

Many types have compound structures; i.e. they are composed of smaller units. Examples include

$$\begin{array}{|l} \hline \text{a man and} \\ \text{a woman} \\ \hline \end{array}
 \quad
 \begin{array}{|l} \hline \text{a food portion } f \text{ and} \\ \text{a child } c \text{ such that } c \\ \text{ate all of } f \\ \hline \end{array}
 \quad
 \begin{array}{|l} \hline \text{a triple } (p, a, j) \text{ where } p \text{ is} \\ \text{a paper, } a \text{ is an author of} \\ p, \text{ and } j \text{ is a journal in} \\ \text{which } p \text{ was published} \\ \hline \end{array}
 \quad (2.10)$$

It is good practice to declare the variables in a “compound type”, as I did in the last two cases of (2.10). In other words, it is preferable to replace the first box above with something like

$$\begin{array}{|l} \hline \text{a man } m \text{ and} \\ \text{a woman } w \\ \hline \end{array}
 \quad \text{or} \quad
 \begin{array}{|l} \hline \text{a pair } (m, w) \\ \text{where } m \text{ is a man} \\ \text{and } w \text{ is a woman} \\ \hline \end{array}$$

so that the variables (m, w) are clear.

Rules of good practice 2.3.1.2. A type is presented as a text box. The text in that box should

- (i) begin with the word “a” or “an”;
- (ii) refer to a distinction made and recognizable by the olog’s author;
- (iii) refer to a distinction for which instances can be documented;
- (iv) declare all variables in a compound structure.

The first, second, and third rules ensure that the class of things represented by each box appears to the author as a well-defined set. The fourth rule encourages good “readability” of arrows, as will be discussed next in Section 2.3.2.

I will not always follow the rules of good practice throughout this document. I think of these rules being followed “in the background” but that I have “nicknamed” various boxes. So \ulcorner Steve \urcorner may stand as a nickname for \ulcorner a thing classified as Steve \urcorner and \ulcorner arginine \urcorner as a nickname for \ulcorner a molecule of arginine \urcorner . However, when pressed, one should always be able to rename each type according to the rules of good practice.

2.3.2 Aspects

An aspect of a thing x is a way of viewing it, a particular way in which x can be regarded or measured. For example, a woman can be regarded as a person; hence “being a person” is an aspect of a woman. A molecule has a molecular mass (say in daltons), so “having a molecular mass” is an aspect of a molecule. In other words, by *aspect* we simply mean

a function. The domain A of the function $f: A \rightarrow B$ is the thing we are measuring, and the codomain is the set of possible “answers” or results of the measurement.

$$\boxed{\text{a woman}} \xrightarrow{\text{is}} \boxed{\text{a person}} \quad (2.11)$$

$$\boxed{\text{a molecule}} \xrightarrow{\text{has as molecular mass (Da)}} \boxed{\text{a positive real number}} \quad (2.12)$$

So for the arrow in (2.11), the domain is the set of women (a set with perhaps 3 billion elements); the codomain is the set of persons (a set with perhaps 6 billion elements). We can imagine drawing an arrow from each dot in the “woman” set to a unique dot in the “person” set, just as in (2.2). No woman points to two different people, nor to zero people — each woman is exactly one person — so the rules for a function are satisfied. Let us now concentrate briefly on the arrow in (2.12). The domain is the set of molecules, the codomain is the set $\mathbb{R}_{>0}$ of positive real numbers. We can imagine drawing an arrow from each dot in the “molecule” set to a single dot in the “positive real number” set. No molecule points to two different masses, nor can a molecule have no mass: each molecule has exactly one mass. Note however that two different molecules can point to the same mass.

2.3.2.1 Invalid aspects

I tried above to clarify what it is that makes an aspect “valid”, namely that it must be a “functional relationship.” In this subsection I will show two arrows which on their face may appear to be aspects, but which on closer inspection are not functional (and hence are not valid as aspects).

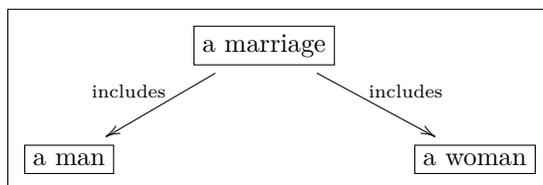
Consider the following two arrows:

$$\boxed{\text{a person}} \xrightarrow{\text{has}} \boxed{\text{a child}} \quad (2.13^*)$$

$$\boxed{\text{a mechanical pencil}} \xrightarrow{\text{uses}} \boxed{\text{a piece of lead}} \quad (2.14^*)$$

A person may have no children or may have more than one child, so the first arrow is invalid: it is not a function. Similarly, if we drew an arrow from each mechanical pencil to each piece of lead it uses, it would not be a function.

Warning 2.3.2.2. The author of an olog has a world-view, some fragment of which is captured in the olog. When person A examines the olog of person B, person A may or may not “agree with it.” For example, person B may have the following olog



which associates to each marriage a man and a woman. Person A may take the position that some marriages involve two men or two women, and thus see B’s olog as “wrong.”

Such disputes are not “problems” with either A’s olog or B’s olog, they are discrepancies between world-views. Hence, throughout this paper, a reader R may see a displayed olog and notice a discrepancy between R’s world-view and my own, but R should not worry that this is a problem. This is not to say that ologs need not follow rules, but instead that the rules are enforced to ensure that an olog is structurally sound, rather than that it “correctly reflects reality,” whatever that may mean.

Consider the aspect $\lceil \text{an object} \rceil \xrightarrow{\text{has}} \lceil \text{a weight} \rceil$. At some point in history, this would have been considered a valid function. Now we know that the same object would have a different weight on the moon than it has on earth. Thus as world-views change, we often need to add more information to our olog. Even the validity of $\lceil \text{an object on earth} \rceil \xrightarrow{\text{has}} \lceil \text{a weight} \rceil$ is questionable. However to build a model we need to choose a level of granularity and try to stay within it, or the whole model evaporates into the nothingness of truth!

Remark 2.3.2.3. In keeping with Warning 2.3.2.2, the arrows (2.13*) and (2.14*) may not be wrong but simply reflect that the author has a strange world-view or a strange vocabulary. Maybe the author believes that every mechanical pencil uses exactly one piece of lead. If this is so, then $\lceil \text{a mechanical pencil} \rceil \xrightarrow{\text{uses}} \lceil \text{a piece of lead} \rceil$ is indeed a valid aspect! Similarly, suppose the author meant to say that each person *was once* a child, or that a person has an inner child. Since every person has one and only one inner child (according to the author), the map $\lceil \text{a person} \rceil \xrightarrow{\text{has as inner child}} \lceil \text{a child} \rceil$ is a valid aspect. We cannot fault the olog if the author has a view, but note that we have changed the name of the label to make his or her intention more explicit.

2.3.2.4 Reading aspects and paths as English phrases

Each arrow (aspect) $X \xrightarrow{f} Y$ can be read by first reading the label on its source box (domain of definition) X , then the label on the arrow f , and finally the label on its target box (set of values) Y . For example, the arrow

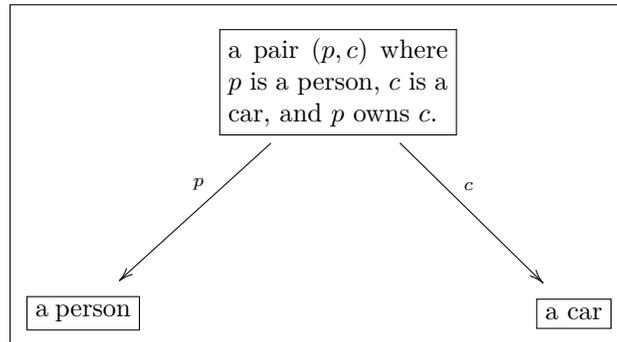
$$\boxed{\text{a book}} \xrightarrow{\text{has as first author}} \boxed{\text{a person}} \quad (2.15)$$

is read “a book has as first author a person”.

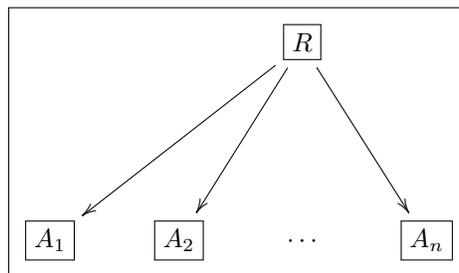
Remark 2.3.2.5. Note that the map in (2.15) is a valid aspect, but that a similarly benign-looking map $\lceil \text{a book} \rceil \xrightarrow{\text{has as author}} \lceil \text{a person} \rceil$ would not be valid, because it is not functional. The authors of an olog must be vigilant about this type of mistake because it is easy to miss and it can corrupt the olog.

Sometimes the label on an arrow can be shortened or dropped altogether if it is obvious from context. We will discuss this more in Section 2.3.3 but here is a common

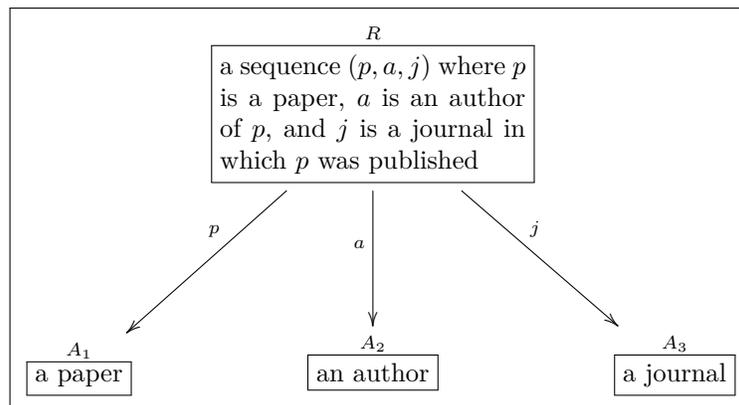
another way to indicate such “non-functional” relationships. In this case it would look like this:



This setup will ensure that everything is properly organized. In general, relationships can involve more than two types, and the general situation looks like this



For example,



Exercise 2.3.2.7. On page 25 we indicate a so-called invalid aspect, namely

$$\boxed{\text{a person}} \xrightarrow{\text{has}} \boxed{\text{a child}} \quad (2.13^*)$$

Create a (valid) olog that captures the parent-child relationship; your olog should still have boxes \lceil a person \rceil and \lceil a child \rceil but may have an additional box. \diamond

Rules of good practice 2.3.2.8. An aspect is presented as a labeled arrow, pointing from a source box to a target box. The arrow text should

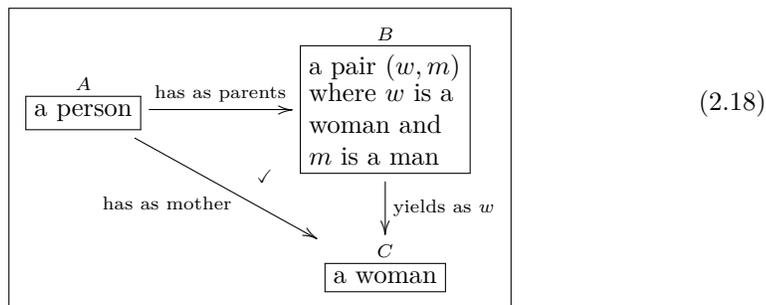
- (i) begin with a verb;
- (ii) yield an English sentence, when the source-box text followed by the arrow text followed by the target-box text is read; and
- (iii) refer to a functional relationship: each instance of the source type should give rise to a specific instance of the target type.

2.3.3 Facts

In this section I will discuss facts, which are simply “path equivalences” in an olog. It is the notion of path equivalences that make category theory so powerful.

A *path* in an olog is a head-to-tail sequence of arrows. That is, any path starts at some box B_0 , then follows an arrow emanating from B_0 (moving in the appropriate direction), at which point it lands at another box B_1 , then follows any arrow emanating from B_1 , etc, eventually landing at a box B_n and stopping there. The number of arrows is the *length* of the path. So a path of length 1 is just an arrow, and a path of length 0 is just a box. We call B_0 the *source* and B_n the *target* of the path.

Given an olog, the author may want to declare that two paths are equivalent. For example consider the two paths from A to C in the olog



We know as English speakers that a woman parent is called a mother, so these two paths $A \rightarrow C$ should be equivalent. A more mathematical way to say this is that the triangle in Olog (2.18) *commutes*. That is, path equivalences are simply commutative diagrams as in Section 2.2. In the example above we concisely say “a woman parent is equivalent to a mother.” We declare this by defining the diagonal map in (2.18) to be *the composition* of the horizontal map and the vertical map.

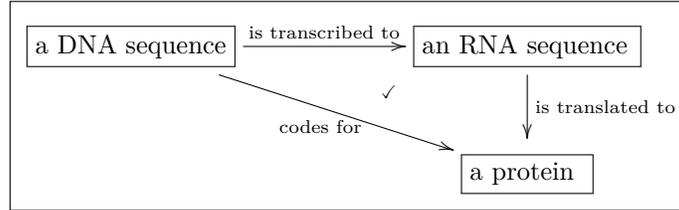
I generally prefer to indicate a commutative diagram by drawing a check-mark, \checkmark , in the region bounded by the two paths, as in Olog (2.18). Sometimes, however, one cannot do this unambiguously on the 2-dimensional page. In such a case I will indicate the commutative diagrams (fact) by writing an equation. For example to say that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h \downarrow & & \downarrow g \\ C & \xrightarrow{i} & D \end{array}$$

commutes, we could either draw a checkmark inside the square or write the equation

$Afg \simeq Ahi$ above it.⁸ Either way, it means that “ f then g ” is equivalent to “ h then i ”.

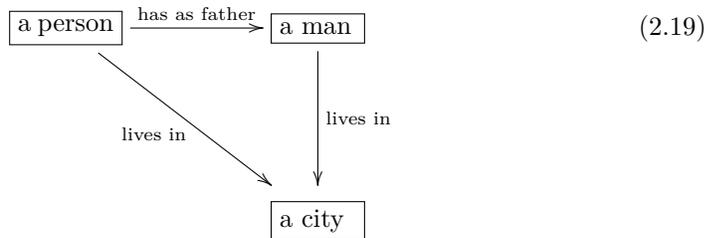
Here is another, more scientific example:



Note how this diagram gives us the established terminology for the various ways in which DNA, RNA, and protein are related in this context.

Exercise 2.3.3.1. Create an olog for human nuclear biological families that includes the concept of person, man, woman, parent, father, mother, and child. Make sure to label all the arrows, and make sure each arrow indicates a valid aspect in the sense of Section 2.3.2.1. Indicate with check-marks (✓) the diagrams that are intended to commute. If the 2-dimensionality of the page prevents a check-mark from being unambiguous, indicate the intended commutativity with an equation. \diamond

Example 2.3.3.2 (Non-commuting diagram). In my conception of the world, the following diagram does not commute:



The non-commutativity of Diagram (2.19) does not imply that, in my conception, no person lives in the same city as his or her father. Rather it implies that, in my conception, it is not the case that *every* person lives in the same city as his or her father.

Exercise 2.3.3.3. Create an olog about a scientific subject, preferably one you think about often. The olog should have at least five boxes, five arrows, and one commutative diagram. \diamond

2.3.3.4 A formula for writing facts as English

Every fact consists of two paths, say P and Q , that are to be declared equivalent. The paths P and Q will necessarily have the same source, say s , and target, say t , but their

⁸We defined function composition on page 2.1.2, but here we’re using a different notation. There we would have said $g \circ f = i \circ h$, which is in the backwards-seeming *classical order*. Category theorists and others often prefer the *diagrammatic order* for writing compositions, which is $f; g = h; i$. For ologs, we follow the latter because it makes for better English sentences, and for the same reason we add the source object to the equation, writing $Afg \simeq Ahi$.

lengths may be different, say m and n respectively.⁹ We draw these paths as

$$\begin{array}{l}
 P : \quad a_0=s \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \xrightarrow{f_3} \dots \xrightarrow{f_{m-1}} a_{m-1} \xrightarrow{f_m} a_m=t \\
 Q : \quad b_0=s \xrightarrow{g_1} b_1 \xrightarrow{g_2} b_2 \xrightarrow{g_3} \dots \xrightarrow{g_{n-1}} b_{n-1} \xrightarrow{g_n} b_n=t
 \end{array} \tag{2.20}$$

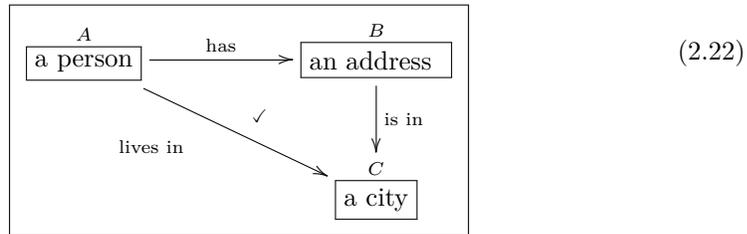
Every part ℓ of an olog (i.e. every box and every arrow) has an associated English phrase, which we write as “ ℓ ”. Using a dummy variable x we can convert a fact into English too. The following general formula is a bit difficult to understand, see Example 2.3.3.5, but here goes. The fact $P \simeq Q$ from (2.20) can be Englishified as follows:

Given x , “ s ”, consider the following. We know that x is “ s ”,
 which “ f_1 ” “ a_1 ”, which “ f_2 ” “ a_2 ”, which ... “ f_{m-1} ” “ a_{m-1} ”, which “ f_m ” “ t ”
 that we’ll call $P(x)$.

We also know that x is “ s ”,
 which “ g_1 ” “ b_1 ”, which “ g_2 ” “ b_2 ”, which ... “ g_{n-1} ” “ b_{n-1} ”, which “ g_n ” “ t ”
 that we’ll call $Q(x)$.

Fact: whenever x is “ s ”, we will have $P(x) = Q(x)$.

Example 2.3.3.5. Consider the olog



To put the fact that Diagram 2.22 commutes into English, we first Englishify the two paths: F = “a person has an address which is in a city” and G = “a person lives in a city”. The source of both is s = “a person” and the target of both is t = “a city”. write:

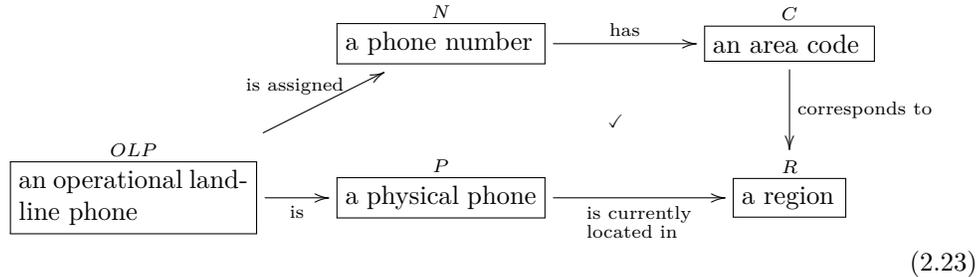
Given x , a person, consider the following. We know that x is a person,
 which has an address, which is in a city
 that we’ll call $P(x)$.

We also know that x is a person,
 which lives in a city
 that we’ll call $Q(x)$.

Fact: whenever x is a person, we will have $P(x) = Q(x)$.

⁹If the source equals the target, $s = t$, then it is possible to have $m = 0$ or $n = 0$, and the ideas below still make sense.

Exercise 2.3.3.6. This olog was taken from [Sp1].



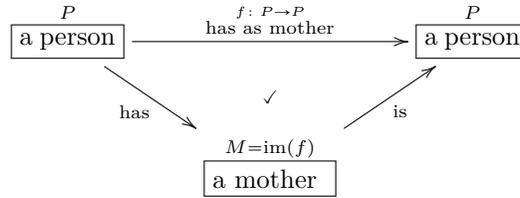
It says that a landline phone is physically located in the region that its phone number is assigned. Translate this fact into English using the formula from 2.21. \diamond

Exercise 2.3.3.7. In the above olog (2.23), suppose that the box "an operational landline phone" is replaced with the box "an operational mobile phone". Would the diagram still commute? \diamond

2.3.3.8 Images

In this section we discuss a specific kind of fact, generated by any aspect. Recall that every function has an image, meaning the subset of elements in the codomain that are "hit" by the function. For example the function $f(x) = 2 * x: \mathbb{Z} \rightarrow \mathbb{Z}$ has as image the set of all even numbers.

Similarly the set of mothers arises as is the image of the "has as mother" function, as shown below



Exercise 2.3.3.9. For each of the following types, write down a function for which it is the image, or say "not clearly an image type"

- "a book"
- "a material that has been fabricated by a process of type T "
- "a bicycle owner"
- "a child"
- "a used book"
- "an inhabited residence"

\diamond

2.4 Products and coproducts

In this section we introduce two concepts that are likely to be familiar, although perhaps not by their category-theoretic names, product and coproduct. Each is an example of a

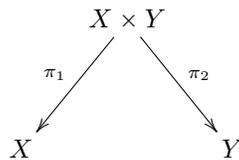
large class of ideas that exist far beyond the realm of sets.

2.4.1 Products

Definition 2.4.1.1. Let X and Y be sets. The *product of X and Y* , denoted $X \times Y$, is defined as the set of ordered pairs (x, y) where $x \in X$ and $y \in Y$. Symbolically,

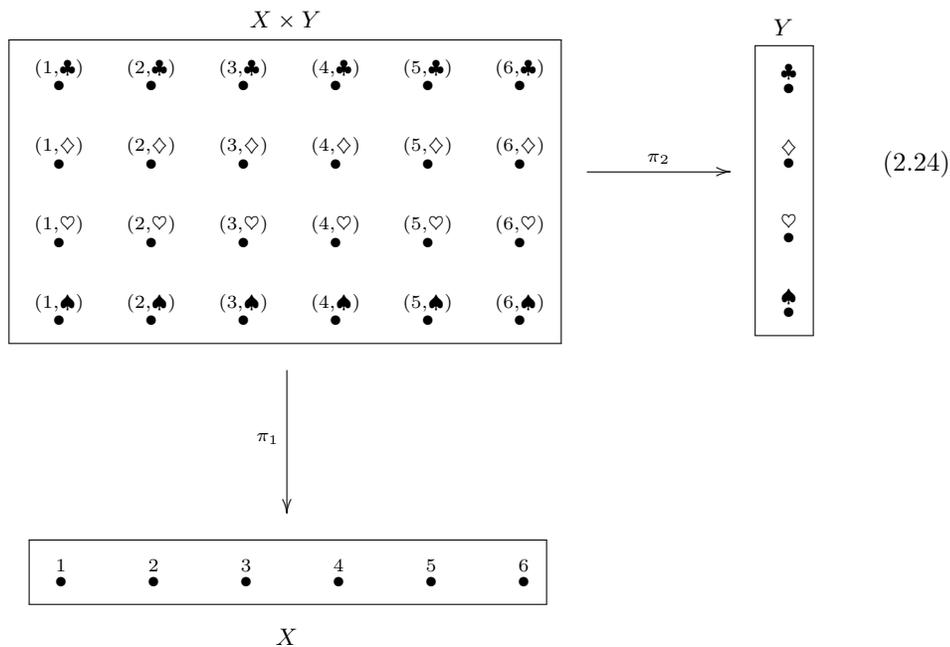
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

There are two natural *projection functions* $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$.



Example 2.4.1.2. [Grid of dots]

Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{\clubsuit, \diamond, \heartsuit, \spadesuit\}$. Then we can draw $X \times Y$ as a 6-by-4 grid of dots, and the projections as projections



Application 2.4.1.3. A traditional (Mendelian) way to predict the genotype of offspring based on the genotype of its parents is by the use of [Punnett squares](#). If F is the set of possible genotypes for the female parent and M is the set of possible genotypes of the male parent, then $F \times M$ is drawn as a square, called a Punnett square, in which every combination is drawn. ◇◇

Exercise 2.4.1.4. How many elements does the set $\{a, b, c, d\} \times \{1, 2, 3\}$ have? ◇

Application 2.4.1.5. Suppose we are conducting experiments about the mechanical properties of materials, as in Application 2.1.2.1. For each material sample we will produce multiple data points in the set $\lceil \text{extension} \rceil \times \lceil \text{force} \rceil \cong \mathbb{R} \times \mathbb{R}$.

◇◇

Remark 2.4.1.6. It is possible to take the product of more than two sets as well. For example, if A, B , and C are sets then $A \times B \times C$ is the set of triples,

$$A \times B \times C := \{(a, b, c) \mid a \in A, b \in B, c \in C\}.$$

This kind of generality is useful in understanding multiple dimensions, e.g. what physicists mean by 10-dimensional space. It comes under the heading of *limits*, which we will see in Section 4.5.3.

Example 2.4.1.7. Let \mathbb{R} be the set of real numbers. By \mathbb{R}^2 we mean $\mathbb{R} \times \mathbb{R}$ (though see Exercise 2.7.2.6). Similarly, for any $n \in \mathbb{N}$, we define \mathbb{R}^n to be the product of n copies of \mathbb{R} .

According to [Pen], Aristotle seems to have conceived of space as something like $S := \mathbb{R}^3$ and of time as something like $T := \mathbb{R}$. Spacetime, had he conceived of it, would probably have been $S \times T \cong \mathbb{R}^4$. He of course did not have access to this kind of abstraction, which was probably due to Descartes.

Exercise 2.4.1.8. Let \mathbb{Z} denote the set of integers, and let $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ denote the addition function and $\cdot: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ denote the multiplication function. Which of the following diagrams commute?

a.)

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} & \xrightarrow{(a,b,c) \mapsto (a \cdot b, a \cdot c)} & \mathbb{Z} \times \mathbb{Z} \\ (a,b,c) \mapsto (a+b,c) \downarrow & & \downarrow (x,y) \mapsto x+y \\ \mathbb{Z} \times \mathbb{Z} & \xrightarrow{(x,y) \mapsto xy} & \mathbb{Z} \end{array}$$

b.)

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{x \mapsto (x,0)} & \mathbb{Z} \times \mathbb{Z} \\ & \searrow \text{id}_{\mathbb{Z}} & \downarrow (a,b) \mapsto a \cdot b \\ & & \mathbb{Z} \end{array}$$

c.)

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{x \mapsto (x,1)} & \mathbb{Z} \times \mathbb{Z} \\ & \searrow \text{id}_{\mathbb{Z}} & \downarrow (a,b) \mapsto a \cdot b \\ & & \mathbb{Z} \end{array}$$

◇

2.4.1.9 Universal property for products

Lemma 2.4.1.10 (Universal property for product). *Let X and Y be sets. For any set A and functions $f: A \rightarrow X$ and $g: A \rightarrow Y$, there exists a unique function $A \rightarrow X \times Y$*

such that the following diagram commutes ¹⁰

$$\begin{array}{ccc}
 & X \times Y & \\
 \pi_1 \swarrow & \uparrow & \searrow \pi_2 \\
 X & \exists! & Y \\
 \forall f \swarrow & \uparrow & \searrow \forall g \\
 & A &
 \end{array}
 \tag{2.25}$$

We might write the unique function as

$$\langle f, g \rangle: A \rightarrow X \times Y.$$

Proof. Suppose given f, g as above. To provide a function $\ell: A \rightarrow X \times Y$ is equivalent to providing an element $\ell(a) \in X \times Y$ for each $a \in A$. We need such a function for which $\pi_1 \circ \ell = f$ and $\pi_2 \circ \ell = g$. An element of $X \times Y$ is an ordered pair (x, y) , and we can use $\ell(a) = (x, y)$ if and only if $x = \pi_1(x, y) = f(a)$ and $y = \pi_2(x, y) = g(a)$. So it is necessary and sufficient to define

$$\langle f, g \rangle(a) := (f(a), g(a))$$

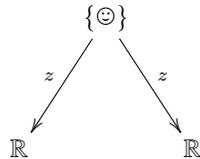
for all $a \in A$.

□

Example 2.4.1.11 (Grid of dots, continued). We need to see the universal property of products as completely intuitive. Recall that if X and Y are sets, say of cardinalities $|X| = m$ and $|Y| = n$ respectively, then $X \times Y$ is an $m \times n$ grid of dots, and it comes with two canonical projections $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$. These allow us to extract from every grid element $z \in X \times Y$ its column $\pi_1(z) \in X$ and its row $\pi_2(z) \in Y$.

Suppose that each person in a classroom picks an element of X and an element of Y . Thus we have functions $f: C \rightarrow X$ and $g: C \rightarrow Y$. But isn't picking a column and a row the same thing as picking an element in the grid? The two functions f and g induce a unique function $C \rightarrow X \times Y$. And how does this function $C \rightarrow X \times Y$ compare with the original functions f and g ? The commutative diagram (2.25) sums up the obvious connection.

Example 2.4.1.12. Let \mathbb{R} be the set of real numbers. The origin in \mathbb{R} is an element of \mathbb{R} . As you showed in Exercise 2.1.2.14, we can view this (or any) element of \mathbb{R} as a function $z: \{\ominus\} \rightarrow \mathbb{R}$, where $\{\ominus\}$ is any set with one element. Our function z “picks out the origin”. Thus we can draw functions



¹⁰The symbol \forall is read “for all”; the symbol \exists is read “there exists”, and the symbol $\exists!$ is read “there exists a unique”. So this diagram is intended to express the idea that for any functions $f: A \rightarrow X$ and $g: A \rightarrow Y$, there exists a unique function $A \rightarrow X \times Y$ for which the two triangles commute.

The universal property for products guarantees a function $\{\odot\} \rightarrow \mathbb{R} \times \mathbb{R}$, which will be the origin in \mathbb{R}^2 .

Remark 2.4.1.13. Given sets X, Y , and A , and functions $f: A \rightarrow X$ and $g: A \rightarrow Y$, there is a unique function $A \rightarrow X \times Y$ that commutes with f and g . We call it *the induced function* $A \rightarrow X \times Y$, meaning the one that arises in light of f and g .

Exercise 2.4.1.14. For every set A there is some nice relationship between the following three sets:

$$\text{Hom}_{\text{Set}}(A, X), \quad \text{Hom}_{\text{Set}}(A, Y), \quad \text{and} \quad \text{Hom}_{\text{Set}}(A, X \times Y).$$

What is it?

Hint: Do not be alarmed: this problem is a bit “recursive” in that you’ll use products in your formula. \diamond

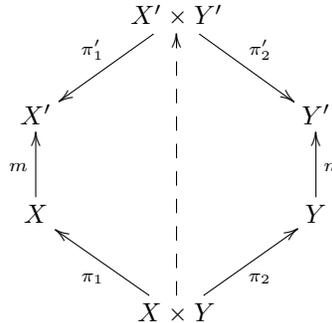
Exercise 2.4.1.15.

- Let X and Y be sets. Construct the “swap map” $s: X \times Y \rightarrow Y \times X$ using only the universal property for products. If $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$ are the projection functions, write s in terms of the symbols “ π_1 ”, “ π_2 ”, “ $(,)$ ”, and “ \circ ”.
- Can you prove that s is an isomorphism using only the universal property for product? \diamond

Example 2.4.1.16. Suppose given sets X, X', Y, Y' and functions $m: X \rightarrow X'$ and $n: Y \rightarrow Y'$. We can use the universal property of products to construct a function $s: X \times Y \rightarrow X' \times Y'$. Here’s how.

The universal property (Lemma 2.4.1.10) says that to get a function from any set A to $X' \times Y'$, we need two functions, namely some $f: A \rightarrow X'$ and some $g: A \rightarrow Y'$. Here $A = X \times Y$.

What we have readily available are the two projections $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$. But we also have $m: X \rightarrow X'$ and $n: Y \rightarrow Y'$. Composing, we set $f := m \circ \pi_1$ and $g := n \circ \pi_2$.



The dotted arrow is often called the *product* of $m: X \rightarrow X'$ and $n: Y \rightarrow Y'$ and is denoted simply by

$$m \times n: X \times Y \rightarrow X' \times Y'.$$

2.4.1.17 Ologging products

Given two objects c, d in an olog, there is a canonical label “ $c \times d$ ” for their product $c \times d$, written in terms of the labels “ c ” and “ d ”. Namely,

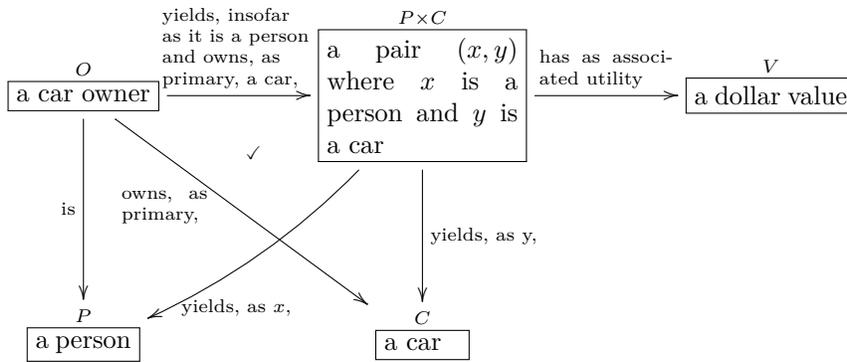
$$“c \times d” := \text{a pair } (x, y) \text{ where } x \text{ is “}c\text{” and } y \text{ is “}d\text{”}.$$

The projections $c \leftarrow c \times d \rightarrow d$ can be labeled “yields, as x ,” and “yields, as y ,” respectively.

Suppose that e is another object and $p: e \rightarrow c$ and $q: e \rightarrow d$ are two arrows. By the universal property of products (Lemma 2.4.1.10), p and q induce a unique arrow $e \rightarrow c \times d$ making the evident diagrams commute. This arrow can be labeled

yields, insofar as it “ p ” “ c ” and “ q ” “ d ”,

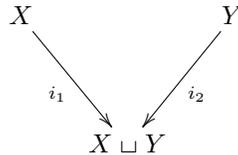
Example 2.4.1.18. Every car owner owns at least one car, but there is no obvious function “a car owner” \rightarrow “a car” because he or she may own more than one. One good choice would be the car that the person drives most often, which we’ll call his or her primary car. Also, given a person and a car, an economist could ask how much utility the person would get out of the car. From all this we can put together the following olog involving products:



2.4.2 Coproducts

Definition 2.4.2.1. Let X and Y be sets. The *coproduct of X and Y* , denoted $X \sqcup Y$, is defined as the “disjoint union” of X and Y , i.e. the set for which an element is either an element of X or an element of Y . If something is an element of both X and Y then we include both copies, and distinguish between them, in $X \sqcup Y$. See Example 2.4.2.2

There are two natural inclusion functions $i_1: X \rightarrow X \sqcup Y$ and $i_2: Y \rightarrow X \sqcup Y$.



Example 2.4.2.2. The coproduct of $X := \{a, b, c, d\}$ and $Y := \{1, 2, 3\}$ is

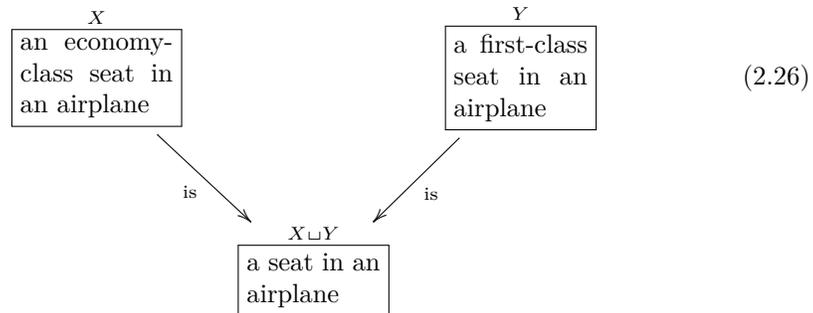
$$X \sqcup Y \cong \{a, b, c, d, 1, 2, 3\}.$$

The coproduct of X and itself is

$$X \sqcup X \cong \{i_1a, i_1b, i_1c, i_1d, i_2a, i_2b, i_2c, i_2d\}$$

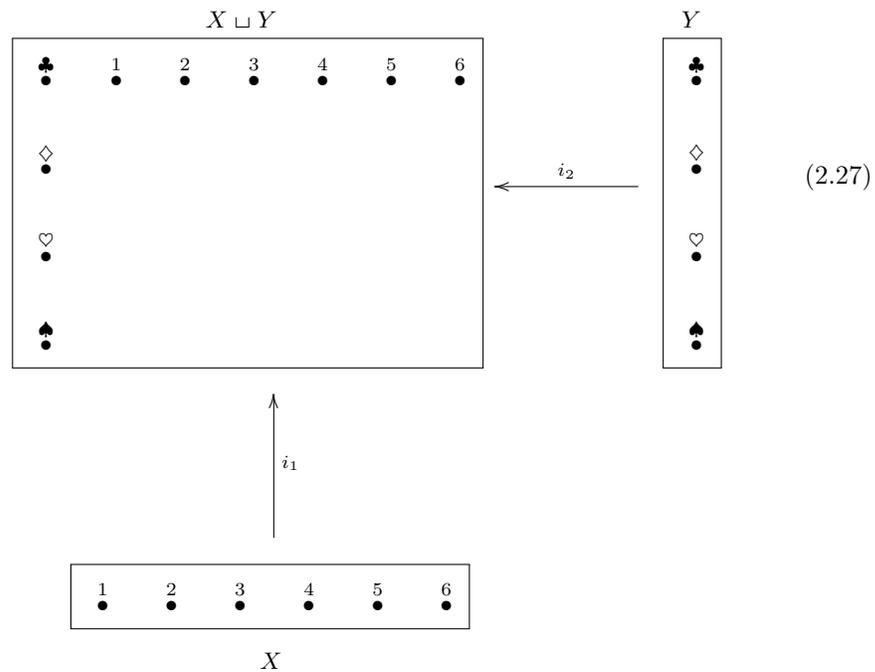
The names of the elements in $X \sqcup Y$ are not so important. What’s important are the inclusion maps i_1, i_2 , which ensure that we know where each element of $X \sqcup Y$ came from.

Example 2.4.2.3 (Airplane seats).



Exercise 2.4.2.4. Would you say that ‘a phone’ is the coproduct of ‘a cellphone’ and ‘a landline phone’? \diamond

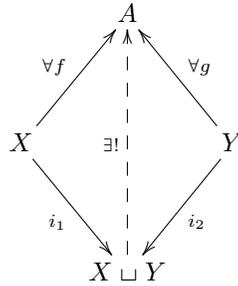
Example 2.4.2.5 (Disjoint union of dots).



2.4.2.6 Universal property for coproducts

Lemma 2.4.2.7 (Universal property for coproduct). *Let X and Y be sets. For any set A and functions $f: X \rightarrow A$ and $g: Y \rightarrow A$, there exists a unique function $X \sqcup Y \rightarrow A$*

such that the following diagram commutes



We might write the unique function as ¹¹

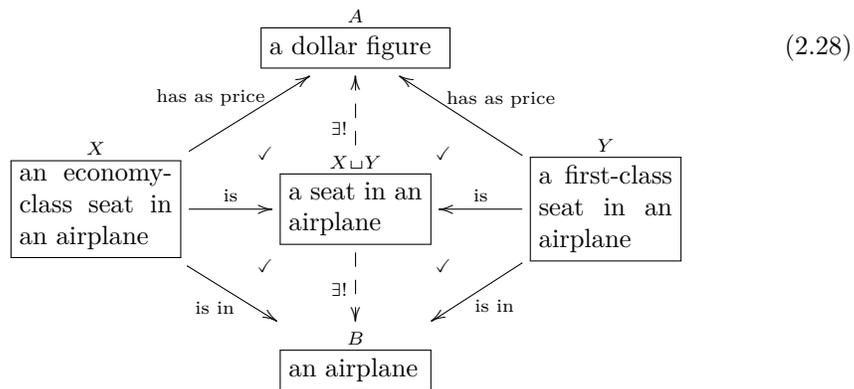
$$\begin{cases} f \\ g \end{cases} : X \sqcup Y \rightarrow A.$$

Proof. Suppose given f, g as above. To provide a function $\ell: X \sqcup Y \rightarrow A$ is equivalent to providing an element $f(m) \in A$ for each $m \in X \sqcup Y$. We need such a function such that $\ell \circ i_1 = f$ and $\ell \circ i_2 = g$. But each element $m \in X \sqcup Y$ is either of the form i_1x or i_2y , and cannot be of both forms. So we assign

$$\begin{cases} f \\ g \end{cases} (m) = \begin{cases} f(x) & \text{if } m = i_1x, \\ g(y) & \text{if } m = i_2y. \end{cases}$$

This assignment is necessary and sufficient to make all relevant diagrams commute. □

Example 2.4.2.8 (Airplane seats, continued). The universal property of coproducts says the following. Any time we have a function $X \rightarrow A$ and a function $Y \rightarrow A$, we get a unique function $X \sqcup Y \rightarrow A$. For example, every economy class seat in an airplane and every first class seat in an airplane is actually *in a particular airplane*. Every economy class seat has a price, as does every first class seat.



The universal property of coproducts formalizes the following intuitively obvious fact:

¹¹We are about to use a two-line symbol, which is a bit unusual. In what follows a certain function $X \sqcup Y \rightarrow A$ is being denoted by the symbol $\begin{cases} f \\ g \end{cases}$.

If we know how economy class seats are priced and we know how first class seats are priced, and if we know that every seat is either economy class or first class, then we automatically know how all seats are priced.

To say it another way (and using the other induced map):

If we keep track of which airplane every economy class seat is in and we keep track of which airplane every first class seat is in, and if we know that every seat is either economy class or first class, then we require no additional tracking for any airplane seat whatsoever.

Application 2.4.2.9 (Piecewise defined curves). In science, curves are often defined or considered piecewise. For example in testing the mechanical properties of a material, we might be interested in various regions of [deformation](#), such as elastic, plastic, or post-fracture. These are three intervals on which the material displays different kinds of properties.

For real numbers $a < b \in \mathbb{R}$, let $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$ denote the closed interval. Given a function $[a, b] \rightarrow \mathbb{R}$ and a function $[c, d] \rightarrow \mathbb{R}$, the universal property of coproducts implies that they extend uniquely to a function $[a, b] \sqcup [c, d] \rightarrow \mathbb{R}$, which will appear as a piecewise defined curve.

Often we are given a curve on $[a, b]$ and another on $[b, c]$, where the two curves agree at the point b . This situation is described by pushouts, which are mild generalizations of coproducts; see Section [2.6.2](#).

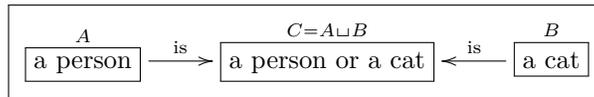
◇

Exercise 2.4.2.10. Write the universal property for coproduct in terms of a relationship between the following three sets:

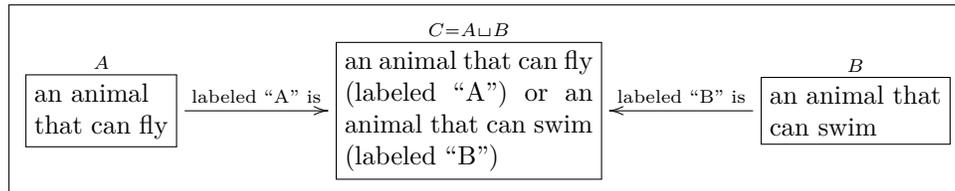
$$\text{Hom}_{\mathbf{Set}}(X, A), \quad \text{Hom}_{\mathbf{Set}}(Y, A), \quad \text{and} \quad \text{Hom}_{\mathbf{Set}}(X \sqcup Y, A).$$

◇

Example 2.4.2.11. In the following olog the types A and B are disjoint, so the coproduct $C = A \sqcup B$ is just the union.



Example 2.4.2.12. In the following olog, A and B are not disjoint, so care must be taken to differentiate common elements.



Since ducks can both swim and fly, each duck is found twice in C , once labeled as a flyer and once labeled as a swimmer. The types A and B are kept disjoint in C , which justifies the name “disjoint union.”

Exercise 2.4.2.13. Understand Example 2.4.2.12 and see if a similar idea would make sense for particles and waves. Make an olog, and choose your wording in accordance with Rules 2.3.1.2. How do photons, which exhibit properties of both waves and particles, fit into the coproduct in your olog?

◇

Exercise 2.4.2.14. Following the section above, “Ologging products” page 36, come up with a naming system for coproducts, the inclusions, and the universal maps. Try it out by making an olog (involving coproducts) discussing the idea that both a .wav file and a .mp3 file can be played on a modern computer. Be careful that your arrows are valid in the sense of Section 2.3.2.1.

◇

2.5 Finite limits in Set

In this section we discuss what are called *limits* of variously-shaped diagrams of sets. We will make all this much more precise when we discuss limits in arbitrary categories in Section 4.5.3.

2.5.1 Pullbacks

Definition 2.5.1.1 (Pullback). Suppose given the diagram of sets and functions below.

$$\begin{array}{ccc} & Y & \\ & \downarrow g & \\ X & \xrightarrow{f} & Z \end{array} \quad (2.29)$$

Its *fiber product* is the set

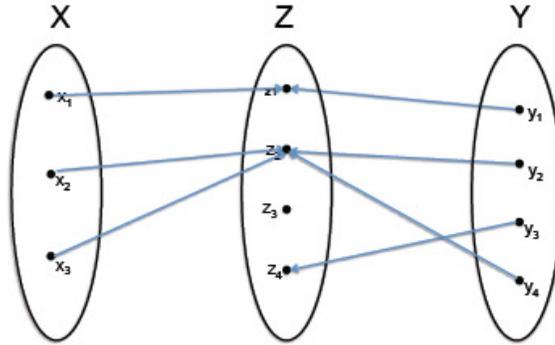
$$X \times_Z Y := \{(x, w, y) \mid f(x) = w = g(y)\}.$$

There are obvious projections $\pi_1: X \times_Z Y \rightarrow X$ and $\pi_2: X \times_Z Y \rightarrow Y$ (e.g. $\pi_2(x, w, y) = y$). Note that if $W = X \times_Z Y$ then the diagram

$$\begin{array}{ccc} W & \xrightarrow{\pi_2} & Y \\ \pi_1 \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array} \quad (2.30)$$

commutes. Given the setup of Diagram 2.29 we define the *pullback of X and Y over Z* to be any set W for which we have an isomorphism $W \xrightarrow{\cong} X \times_Z Y$. The corner symbol \lrcorner in Diagram 2.30 indicates that W is the pullback.

Exercise 2.5.1.2. Let X, Y, Z be as drawn and $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ the indicated functions.



What is the pullback of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$? ◇

Exercise 2.5.1.3.

- a.) Draw a set X with five elements and a set Y with three elements. Color each element of X and each element of Y either red, blue, or yellow,¹² and do so in a “random-looking” way. Considering your coloring of X as a function $X \rightarrow C$, where $C = \{\text{red, blue, yellow}\}$, and similarly obtaining a function $Y \rightarrow C$, draw the fiber product $X \times_C Y$. Make sure it is colored appropriately.
- b.) The universal property for products guarantees a function $X \times_C Y \rightarrow X \times Y$, which I can tell you will be an injection. This means that the drawing you made of the fiber product can be imbedded into the 5×3 grid; please draw the grid and indicate this subset.

◇

Remark 2.5.1.4. Some may prefer to denote this fiber product by $f \times_Z g$ rather than $X \times_Z Y$. The former is mathematically better notation, but human-readability is often enhanced by the latter, which is also more common in the literature. We use whichever is more convenient.

Exercise 2.5.1.5.

- a.) Suppose that $Y = \emptyset$; what can you say about $X \times_Z Y$?
- b.) Suppose now that Y is any set but that Z has exactly one element; what can you say about $X \times_Z Y$?

◇

Exercise 2.5.1.6. Let $S = \mathbb{R}^3, T = \mathbb{R}$, and think of them as (Aristotelian) space and time, with the origin in $S \times T$ given by the center of mass of MIT at the time of its founding. Let $Y = S \times T$ and let $g_1: Y \rightarrow S$ be one projection and $g_2: Y \rightarrow T$ the other projection. Let $X = \{\odot\}$ be a set with one element and let $f_1: X \rightarrow S$ and $f_2: X \rightarrow T$ be given by the origin in both cases.

- a.) What are the fiber products W_1 and W_2 :

$$\begin{array}{ccc}
 W_1 & \longrightarrow & Y \\
 \downarrow & \lrcorner & \downarrow g_1 \\
 X & \xrightarrow{f_1} & S
 \end{array}
 \qquad
 \begin{array}{ccc}
 W_2 & \longrightarrow & Y \\
 \downarrow & \lrcorner & \downarrow g_2 \\
 X & \xrightarrow{f_2} & T
 \end{array}$$

¹²You can use shadings rather than coloring, if coloring would be annoying.

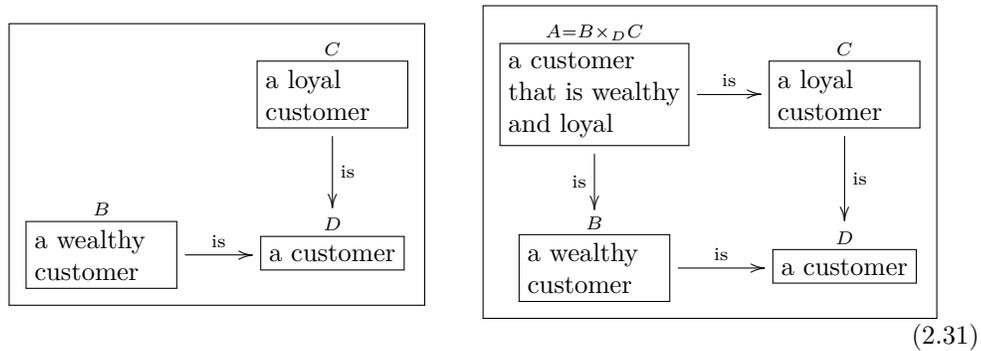
b.) Interpret these sets in terms of the center of mass of MIT at the time of its founding.

◇

2.5.1.7 Using pullbacks to define new ideas from old

In this section we will see that the fiber product of a diagram can serve to define a new concept. For example, in (2.33) we define what it means for a cellphone to have a bad battery, in terms of the length of time for which it remains charged. By being explicit, we reduce the chance of misunderstandings between different groups of people. This can be useful in situations like audits and those in which one is trying to reuse or understand data gathered by others.

Example 2.5.1.8. Consider the following two ologs. The one on the right is the pullback of the one on the left.

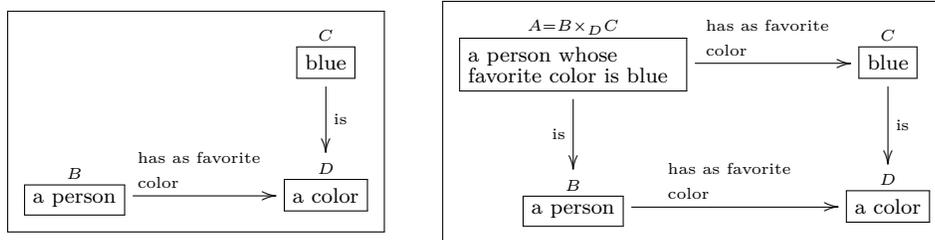


Check from Definition 2.5.1.1 that the label, “a customer that is wealthy and loyal”, is fair and straightforward as a label for the fiber product $A = B \times_D C$, given the labels on B, C , and D .

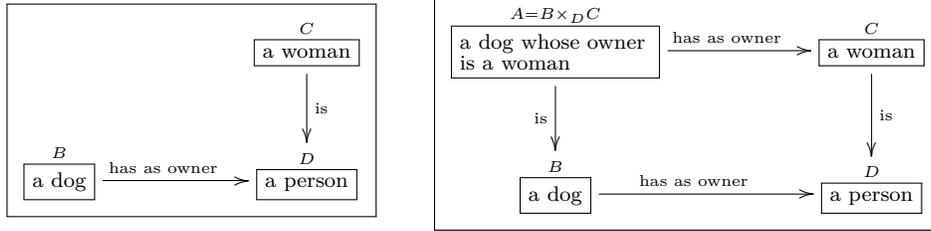
Remark 2.5.1.9. Note that in Diagram (2.31) the top-left box could have been (non-canonically named) “a good customer”. If it was taken to be the fiber product, then the author would be effectively *defining* a good customer to be one that is wealthy and loyal.

Exercise 2.5.1.10. For each of the following, an author has proposed that the diagram on the right is a pullback. Do you think their labels are appropriate or misleading; that is, is the label on the upper-left box reasonable given the rest of the olog, or is it suspect in some way?

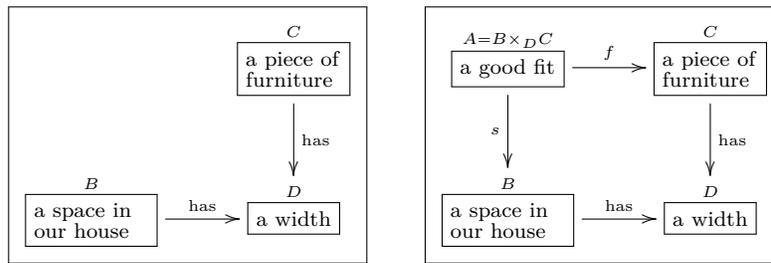
a.)



b.)



c.)



◇

Exercise 2.5.1.11.

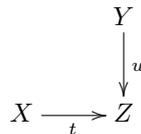
- a.) Consider your olog from Exercise 2.3.3.1. Are any of the commutative squares there actually pullback squares?
- b.) Now use ologs with products and pullbacks to define what a brother is and what a sister is (again in a human biological nuclear family), in terms of types such as «an offspring of mating pair (a, b) », «a person», «a male person», «a female person», and so on.

◇

Definition 2.5.1.12 (Preimage). Let $f: X \rightarrow Y$ be a function and $y \in Y$ an element. The *preimage of y under f* , denoted $f^{-1}(y)$, is the subset $f^{-1}(y) := \{x \in X \mid f(x) = y\}$. If $Y' \subseteq Y$ is any subset, the *preimage of Y' under f* , denoted $f^{-1}(Y')$, is the subset $f^{-1}(Y') = \{x \in X \mid f(x) \in Y'\}$.

Exercise 2.5.1.13. Let $f: X \rightarrow Y$ be a function and $y \in Y$ an element. Draw a pullback diagram in which the fiber product is isomorphic to the preimage $f^{-1}(y)$. ◇

Lemma 2.5.1.14 (Universal property for pullback). Suppose given the diagram of sets and functions as below.



For any set A and commutative solid arrow diagram as below (i.e. functions $f: A \rightarrow X$ and $g: A \rightarrow Y$ such that $t \circ f = u \circ g$),

$$\begin{array}{ccc}
 & X \times_Z Y & \\
 \pi_1 \swarrow & \uparrow \exists! & \searrow \pi_2 \\
 & A & \\
 \forall f \swarrow & & \searrow \forall g \\
 X & & Y \\
 t \searrow & & \swarrow u \\
 & Z &
 \end{array} \tag{2.32}$$

there exists a unique arrow $\langle f, f \rangle_Z: A \rightarrow X \times_Z Y$ making everything commute, i.e.

$$f = \pi_1 \circ \langle f, f \rangle_Z \quad \text{and} \quad g = \pi_2 \circ \langle f, f \rangle_Z.$$

Exercise 2.5.1.15. Create an olog whose underlying shape is a commutative square. Now add the fiber product so that the shape is the same as that of Diagram (2.32). Assign English labels to the projections π_1, π_2 and to the dotted map $A \xrightarrow{\langle f, f \rangle_Z} X \times_Z Y$, such that these labels are as canonical as possible. \diamond

2.5.1.16 Pasting diagrams for pullback

Consider the diagram drawn below, which includes a left-hand square, a right-hand square, and a big rectangle.

$$\begin{array}{ccccc}
 A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \\
 \downarrow i & \lrcorner & \downarrow j & \lrcorner & \downarrow k \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C
 \end{array}$$

The right-hand square has a corner symbol indicating that $B' \cong B \times_C C'$ is a pullback. But the corner symbol on the left is ambiguous; it might be indicating that the left-hand square is a pullback, or it might be indicating that the big rectangle is a pullback. It turns out that if $B' \cong B \times_C C'$ then it is not ambiguous because the left-hand square is a pullback if and only if the big rectangle is.

Proposition 2.5.1.17. Consider the diagram drawn below

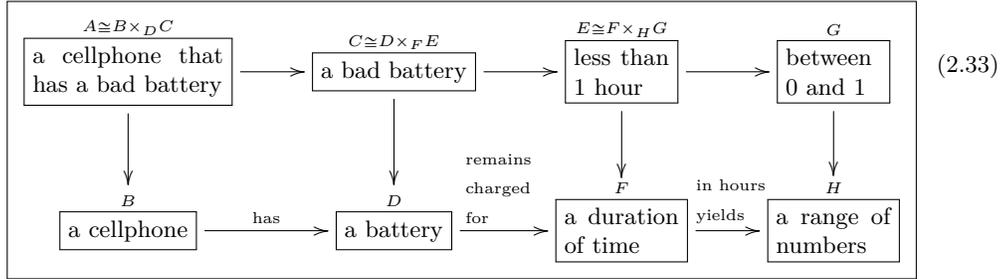
$$\begin{array}{ccc}
 & B' & \xrightarrow{g'} & C' \\
 & \downarrow j & \lrcorner & \downarrow k \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C
 \end{array}$$

where $B' \cong B \times_C C'$ is a pullback. Then there is an isomorphism $A \times_B B' \cong A \times_C C'$. Said another way,

$$A \times_B (B \times_C C') \cong A \times_C C'.$$

Proof. We first provide a map $\phi: A \times_B (B \times_C C') \rightarrow A \times_C C'$. An element of $A \times_B (B \times_C C')$ is of the form $(a, b, (b, c, c'))$ such that $f(a) = b, g(b) = c$ and $k(c') = c$. But this implies that $g \circ f(a) = c = k(c')$ so we put $\phi(a, b, (b, c, c')) := (a, c, c') \in A \times_C C'$. Now we provide a proposed inverse, $\psi: A \times_C C' \rightarrow A \times_B (B \times_C C')$. Given (a, c, c') with $g \circ f(a) = c = k(c')$, let $b = f(a)$ and note that (b, c, c') is an element of $B \times_C C'$. So we can define $\psi(a, c, c') = (a, b, (b, c, c'))$. It is easy to see that ϕ and ψ are inverse. □

Proposition 2.5.1.17 can be useful in authoring ologs. For example, the type \ulcorner a cellphone that has a bad battery \urcorner is vague, but we can lay out precisely what it means using pullbacks:



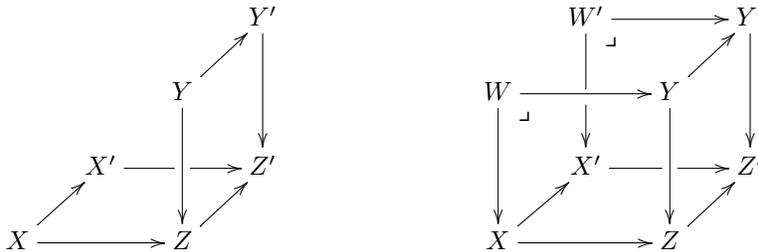
The category-theoretic fact described above says that since $A \cong B \times_D C$ and $C \cong D \times_F E$, it follows that $A \cong B \times_F E$. That is, we can deduce the definition “a cellphone that has a bad battery is defined as a cellphone that has a battery which remains charged for less than one hour.”

Exercise 2.5.1.18.

- a.) Create an olog that defines two people to be “of approximately the same height” if and only if their height difference is less than half an inch, using a pullback. Your olog can include the box \ulcorner a real number x such that $-0.5 < x < 0.5\urcorner$.
- b.) In the same olog, make a box for those people whose height is approximately the same as a person named “The Virgin Mary”. You may need to use images, as in Section 2.3.3.8.

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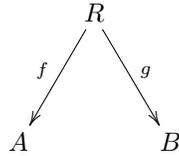
Exercise 2.5.1.19. Consider the diagram on the left below, where both squares commute.



Let $W = X \times_Z Y$ and $W' = X' \times_{Z'} Y'$, and form the diagram to the right. Use the universal property of fiber products to construct a map $W \rightarrow W'$ such that all squares commute. ◇

2.5.2 Spans, experiments, and matrices

Definition 2.5.2.1. Given sets A and B , a *span on A and B* is a set R together with functions $f: R \rightarrow A$ and $g: R \rightarrow B$.

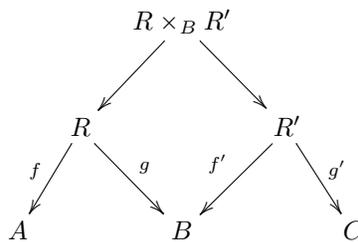


Application 2.5.2.2. Think of A and B as observables and R as a set of experiments performed on these two variables. For example, let's say T is the set of possible temperatures of a gas in a fixed container and let's say P is the set of possible pressures of the gas. We perform 1000 experiments in which we change and record the temperature and we simultaneously also record the pressure; this is a span $T \xleftarrow{f} E \xrightarrow{g} P$. The results might look like this:

Experiment		
ID	Temperature	Pressure
1	100	72
2	100	73
3	100	72
4	200	140
5	200	138
6	200	141
\vdots	\vdots	\vdots

◇◇

Definition 2.5.2.3. Let A, B , and C be sets, and let $A \xleftarrow{f} R \xrightarrow{g} B$ and $B \xleftarrow{f'} R' \xrightarrow{g'} C$ be spans. Their *composite span* is given by the fiber product $R \times_B R'$ as in the diagram below:



Application 2.5.2.4. Let's look back at our lab's experiment from Application 2.5.2.2, which resulted in a span $T \xleftarrow{f} E \xrightarrow{g} P$. Suppose we notice that something looks a little wrong. The pressure should be linear in the temperature but it doesn't appear to be. We hypothesize that the volume of the container is increasing under pressure. We look up this container online and see that experiments have been done to measure the volume as the interior pressure changes. The data has generously been made available online, which gives us a span $P \xleftarrow{f'} E' \xrightarrow{g'} V$.

The composite of our lab's span with the online data span yields a span $T \leftarrow E'' \rightarrow V$, where $E'' := E \times_P E'$. What information does this span give us? In explaining it, one

might say “whenever an experiment in our lab yielded the same pressure as one they recorded, let’s call that a data point. Every data point has an associated temperature (from our lab) and an associated volume (from their experiment). This is the best we can do.”

The information we get this way might be seen by some as unscientific, but it certainly is the kind of information people use in business and in every day life calculation—we get our data from multiple sources and put it together. Moreover, it is scientific in the sense that it is reproducible. The way we obtained our T - V data is completely transparent.

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We can relate spans to matrices of natural numbers, and see a natural “categorification” of matrix addition and matrix multiplication. If our spans come from experiments as in Applications 2.5.2.2 and 2.5.2.4 the matrices involved will look like huge but sparse matrices. Let’s go through that.

Let A and B be sets and let $A \leftarrow R \rightarrow B$ be a span. By the universal property of products, we have a unique map $R \xrightarrow{p} A \times B$.

We make a matrix of natural numbers out of this data as follows. The set of rows is A , the set of columns is B . For elements $a \in A$ and $b \in B$, the (a, b) -entry is the cardinality of its preimage, $|p^{-1}(a, b)|$, i.e. the number of elements in R that are sent by p to (a, b) .

Suppose we are given two (A, B) -spans, i.e. $A \leftarrow R \rightarrow B$ and $A \leftarrow R' \rightarrow B$; we might think of these as having the same *dimensions*, i.e. they are both $|A| \times |B|$ -matrices. We can take the disjoint union $R \sqcup R'$ and by the universal property of coproducts we have a unique span $A \leftarrow R \sqcup R' \rightarrow B$ making the requisite diagram commute.¹³ The matrix corresponding to this new span will be the sum of the matrices corresponding to the two previous spans out of which it was made.

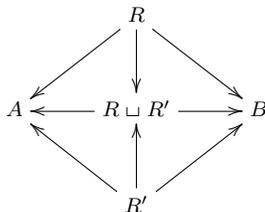
Given a span $A \leftarrow R \rightarrow B$ and a span $B \leftarrow S \rightarrow C$, the composite span can be formed as in Definition 2.5.2.3. It will correspond to the usual multiplication of matrices.

Construction 2.5.2.5. Given a span $A \xleftarrow{f} R \xrightarrow{g} B$, one can draw a *bipartite graph* with each element of A drawn as a dot on the left, each element of B drawn as a dot on the right, and each element $r \in R$ drawn as an arrow connecting vertex $f(r)$ on the left to vertex $g(r)$ on the right.

Exercise 2.5.2.6.

- Draw the bipartite graph (as in Construction 2.5.2.5) corresponding to the span $T \xleftarrow{f} E \xrightarrow{g} P$ in Application 2.5.2.2.
- Now make up your own span $P \xleftarrow{f'} E' \xrightarrow{g'} V$ and draw it. Finally, draw the composite span below.
- Can you say how the composite span graph relates to the graphs of its factors?

¹³



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2.5.3 Equalizers and terminal objects

Definition 2.5.3.1. Suppose given two parallel arrows

$$X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} Y. \quad Eq(f, g) \xrightarrow{p} X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} Y \quad (2.34)$$

The *equalizer of f and g* is the commutative diagram as to the right in (2.34), where we define

$$Eq(f, g) := \{x \in X \mid f(x) = g(x)\}$$

and where p is the canonical inclusion.

Example 2.5.3.2. Suppose one has designed an experiment to test a theoretical prediction. The question becomes, “when does the theory match the experiment?” The answer is given by the equalizer of the following diagram:

$$\boxed{\text{an input}} \begin{array}{c} \xrightarrow{\text{should, according to theory, yield}} \\ \xrightarrow{\text{according to experiment yields}} \end{array} \boxed{\text{an output}}$$

The equalizer is the set of all inputs for which the theory and the experiment yield the same output.

Exercise 2.5.3.3. Come up with an olog that uses equalizers in a reasonably interesting way. Alternatively, use an equalizer to specify those published authors who have published exactly one paper. Hint: find a function from authors to papers; then find another. ◇

Exercise 2.5.3.4. Find a universal property enjoyed by the equalizer of two arrows, and present it in the style of Lemmas 2.4.1.10, 2.4.2.7, and 2.5.1.14. ◇

Exercise 2.5.3.5.

- A terminal set is a set S such that for every set X , there exists a unique function $X \rightarrow S$. Find a terminal set.
- Do you think that the notion *terminal set* belongs in this section (Section 2.5)? How so? If products, pullbacks, and equalizers are all limits, what do limits have in common?

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2.6 Finite colimits in Set

This section will parallel Section 2.5—I will introduce several types of finite colimits and hope that this gives the reader some intuition about them, without formally defining them yet. Before doing so, I must define equivalence relations and quotients.

2.6.1 Background: equivalence relations

Definition 2.6.1.1 (Equivalence relations and equivalence classes). Let X be a set. An *equivalence relation on X* is a subset $R \subseteq X \times X$ satisfying the following properties for all $x, y, z \in X$:

Reflexivity: $(x, x) \in R$;

Symmetry: $(x, y) \in R$ if and only if $(y, x) \in R$; and

Transitivity: if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

If R is an equivalence relation, we often write $x \sim_R y$, or simply $x \sim y$, to mean $(x, y) \in R$. For convenience we may refer to the equivalence relation by the symbol \sim , saying that \sim is an equivalence relation on X .

An *equivalence class of \sim* is a subset $A \subseteq X$ such that

- A is nonempty, $A \neq \emptyset$;
- if $x \in A$ and $x' \in A$, then $x \sim x'$; and
- if $x \in A$ and $x \sim y$, then $y \in A$.

Suppose that \sim is an equivalence relation on X . The *quotient of X by \sim* , denoted X/\sim is the set of equivalence classes of \sim .

Example 2.6.1.2. Let \mathbb{Z} denote the set of integers. Define a relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ by

$$R = \{(x, y) \mid \exists n \in \mathbb{Z} \text{ such that } x + 7n = y\}.$$

Then R is an equivalence relation because $x + 7 * 0 = x$ (reflexivity); $x + 7 * n = y$ if and only if $y + 7 * (-n) = x$ (symmetry); and $x + 7n = y$ and $y + 7m = z$ together imply that $x + 7(m + n) = z$ (transitivity).

Exercise 2.6.1.3. Let X be the set of people on earth; define a binary relation $R \subseteq X \times X$ on X as follows. For a pair (x, y) of people, say $(x, y) \in R$ if x spends a lot of time thinking about y .

- a.) Is this relation reflexive?
- b.) Is it symmetric?
- c.) Is it transitive?

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Example 2.6.1.4 (Partitions). An equivalence relation on a set X can be thought of as a way of partitioning X . A *partition of X* consists of a set I , called *the set of parts*, and for every element $i \in I$ a subset $X_i \subseteq X$ such that two properties hold:

- every element $x \in X$ is in some part (i.e. for all $x \in X$ there exists $i \in I$ such that $x \in X_i$); and
- no element can be found in two different parts (i.e. if $x \in X_i$ and $x \in X_j$ then $i = j$).

Given a partition of X , we define an equivalence relation \sim on X by saying $x \sim x'$ if x and x' are in the same part (i.e. if there exists $i \in I$ such that $x, x' \in X_i$). The parts become the equivalence classes of this relation. Conversely, given an equivalence relation, one makes a partition on X by taking I to be the set of equivalence classes and for each $i \in I$ letting X_i be the elements in that equivalence class.

Exercise 2.6.1.5. Let X and B be sets and let $f: X \rightarrow B$ be a function. Define a subset $R \subseteq X \times X$ by

$$R = \{(x, y) \mid f(x) = f(y)\}.$$

- a.) Is R an equivalence relation?
- b.) Are all equivalence relations on X obtainable in this way (as the fibers of some function having domain X)?
- c.) Does this viewpoint on equivalence classes relate to that of Example 2.6.1.4?

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Exercise 2.6.1.6. Take a set I of sets; i.e. suppose that for each element $i \in I$ you are given a set X_i . For every two elements $i, j \in I$ say that $i \sim j$ if X_i and X_j are isomorphic. Is this relation an equivalence relation on I ?

◇

Lemma 2.6.1.7 (Generating equivalence relations). *Let X be a set and $R \subseteq X \times X$ a subset. There exists a relation $S \subseteq X \times X$ such that*

- S is an equivalence relation,
- $R \subseteq S$, and
- for any equivalence relation S' such that $R \subseteq S'$, we have $S \subseteq S'$.

The relation S' will be called the equivalence relation generated by R .

Proof. Let L_R be the set of all equivalence relations on X that contain R ; in other words, each element $\ell \in L_R$ is an equivalence relation, $\ell \subseteq X \times X$. The set L_R is non-empty because $X \times X \subseteq X \times X$ is an equivalence relation. Let S denote the set of pairs $(x_1, x_2) \in X \times X$ that appear in every element of L_R . Note that $R \subseteq S$ by definition. We need only show that S is an equivalence relation.

It is clearly reflexive, because R is. If $(x, y) \in S$ then $(x, y) \in \ell$ for all $\ell \in L_R$. But since each ℓ is an equivalence relation, $(y, x) \in \ell$ too, so $(y, x) \in S$. This shows that S is symmetric. The proof that it is transitive is similar: if $(x, y) \in S$ and $(y, z) \in S$ then they are both in each ℓ which puts (x, z) in each ℓ , which puts it in S .

□

Remark 2.6.1.8. Let X be a set and $R \subseteq X \times X$ a relation. The proof of Lemma 2.6.1.7 has the benefit of working even if $|X| \geq \infty$, but it has the cost that it is not very intuitive, nor useful in practice when X is finite. The intuitive way to think about the idea of equivalence relation generated by R is as follows.

1. First add to R what is demanded by reflexivity, $R_1 := R \cup \{(x, x) \mid x \in X\}$.
2. Then add to R what is demanded by symmetry, $R_2 := R_1 \cup \{(x, y) \mid (y, x) \in R_1\}$.
3. Finally, add to R what is demanded by transitivity,

$$S = R_2 \cup \{(x, z) \mid (x, y) \in R_2, \text{ and } (y, z) \in R_2\}.$$

Exercise 2.6.1.9. Consider the set \mathbb{R} of real numbers. Draw the coordinate plane $\mathbb{R} \times \mathbb{R}$, give it coordinates x and y . A binary relation on \mathbb{R} is a subset $S \subseteq \mathbb{R} \times \mathbb{R}$, which can be drawn as a set of points in the plane.

- a.) Draw the relation $\{(x, y) \mid y = x^2\}$.
- b.) Draw the relation $\{(x, y) \mid y \geq x^2\}$.
- c.) Let S_0 be the equivalence relation on \mathbb{R} generated (in the sense of Lemma 2.6.1.7) by the empty set. Draw S as a subset of the plane.
- d.) Consider the equivalence relation S_1 generated by $\{(1, 2), (1, 3)\}$. Draw S_1 in the plane. Highlight the equivalence class containing $(1, 2)$.
- e.) The reflexivity property and the symmetry property have pleasing visualizations in $\mathbb{R} \times \mathbb{R}$; what are they?
- f.) Is there a nice heuristic for visualizing the transitivity property?

◇

Exercise 2.6.1.10. Consider the binary relation $R = \{(n, n + 1) \mid n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$.

- a.) What is the equivalence relation generated by R ?
- b.) How many equivalence classes are there?

◇

Exercise 2.6.1.11. Suppose N is a network (or graph). Let X be the nodes of the network, and let $R \subseteq X \times X$ denote the relation such that $(x, y) \in R$ iff there exists an arrow connecting x to y .¹⁴

- a.) What is the equivalence relation \sim generated by R ?
- b.) What is the quotient X/\sim ?

◇

2.6.2 Pushouts

Definition 2.6.2.1 (Pushout). Suppose given the diagram of sets and functions below:

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \\ & & Y \end{array} \quad (2.35)$$

Its *fiber sum*, denoted $X \sqcup_W Y$, is defined as the quotient of $X \sqcup W \sqcup Y$ by the equivalence relation \sim generated by $w \sim f(w)$ and $w \sim g(w)$ for all $w \in W$.

$$X \sqcup_W Y := (X \sqcup W \sqcup Y) / \sim \quad \text{where } \forall w \in W, \quad w \sim f(w) \quad \text{and} \quad w \sim g(w).$$

¹⁴The word *iff* means “if and only if”. In this case we are saying that the pair (x, y) is in R if and only if there exists an arrow connecting x and y .

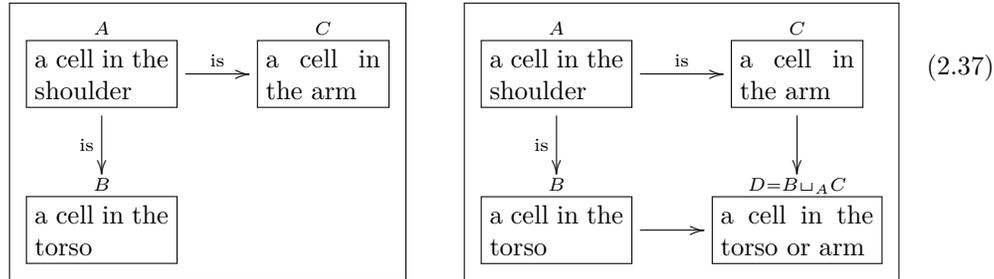
There are obvious inclusions $i_1: X \rightarrow X \sqcup_W Y$ and $i_2: Y \rightarrow X \sqcup_W Y$.¹⁵ Note that if $Z = X \sqcup_W Y$ then the diagram

$$\begin{array}{ccc}
 W & \xrightarrow{g} & Y \\
 f \downarrow & \lrcorner & \downarrow i_2 \\
 X & \xrightarrow{i_1} & Z
 \end{array} \tag{2.36}$$

commutes. Given the setup of Diagram 2.35 we define the *pushout of X and Y over W* to be any set Z for which we have an isomorphism $Z \xrightarrow{\cong} X \sqcup_W Y$. The corner symbol \lrcorner in Diagram 2.36 indicates that Z is the pushout.

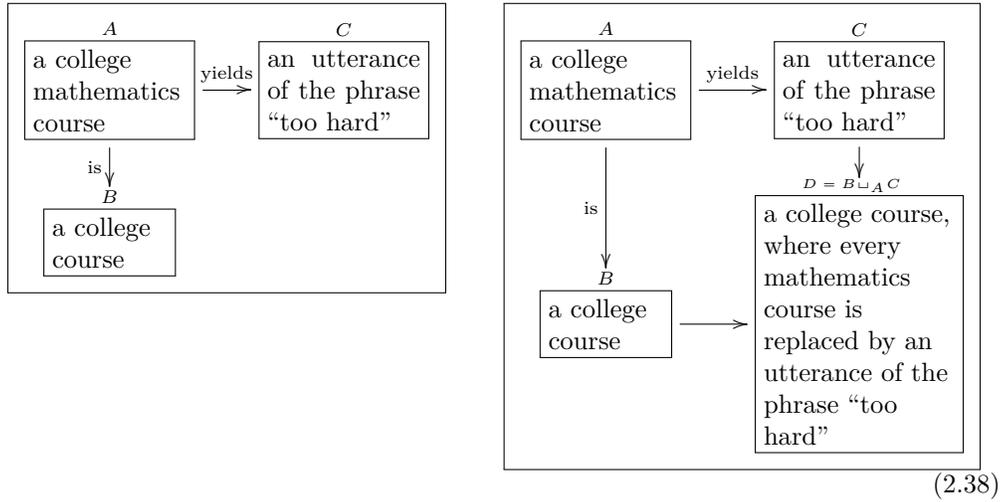
Example 2.6.2.2. Let $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ be the set of numbers between 0 and 1, inclusive, let $Y = \{y \in \mathbb{R} \mid 1 \leq y \leq 2\}$ by the set of numbers between 1 and 2, inclusive, and let $W = \{1\}$. Then the pushout $X \xleftarrow{f} W \xrightarrow{g} Y$, where f and g are the “obvious” functions ($1 \mapsto 1$) is $X \sqcup_W Y \cong \{z \in \mathbb{R} \mid 0 \leq z \leq 2\}$, as expected. When we eventually get to general colimits, one can check that the whole real line can be made by patching together intervals in this way.

Example 2.6.2.3 (Pushout). In each example below, the diagram to the right is intended to be a pushout of the diagram to the left. The new object, D , is the union of B and C , but instances of A are equated to their B and C aspects. This will be discussed after the two diagrams.



In the left-hand olog (2.37, the two arrows are inclusions: the author considers every cell in the shoulder to be both in the arm and in the torso. The pushout is then just the union, where cells in the shoulder are not double-counted.

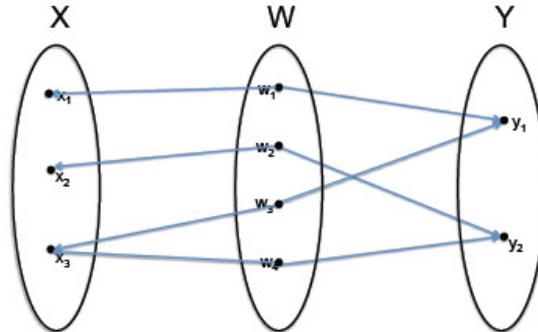
¹⁵Note that our term inclusions is not too good, because it seems to suggest that i_1 and i_2 are injective (see Definition 2.7.5.1) and this is not always the case.



In Olog (2.37), the shoulder is seen as part of the arm and part of the torso. When taking the union of these two parts, we do not want to “double-count” the shoulder (as would be done in the coproduct $B \sqcup C$, see Example 2.4.2.12). Thus we create a new type A for cells in the shoulder, which are considered the same whether viewed as cells in the arm or cells in the torso. In general, if one wishes to take two things and glue them together, with A as the glue and with B and C as the two things to be glued, the union is the pushout $B \sqcup_A C$. (A nice image of this can be seen in the setting of topological spaces, see Example 4.5.3.30.)

In Olog (2.38), if every mathematics course is simply “too hard,” then when reading off a list of courses, each math course will not be read aloud but simply read as “too hard.” To form D we begin by taking the union of B and C , and then we consider everything in A to be the same whether one looks at it as a course or as the phrase “too hard.” The math courses are all blurred together as one thing. Thus we see that the power to equate different things can be exercised with pushouts.

Exercise 2.6.2.4. Let W, X, Y be as drawn and $f: W \rightarrow X$ and $g: W \rightarrow Y$ the indicated functions.



The pushout of the diagram $X \xleftarrow{f} W \xrightarrow{g} Y$ is a set P . Write down the cardinality of $P \cong \underline{n}$ as a natural number $n \in \mathbb{N}$. \diamond

Exercise 2.6.2.5. Suppose that $W = \emptyset$; what can you say about $X \sqcup_W Z$? \diamond

Exercise 2.6.2.6. Let $W := \mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers, let $X = \mathbb{Z}$ denote the set of integers, and let $Y = \{\odot\}$ denote a one-element set. Define $f: W \rightarrow X$ by $f(w) = -(w + 1)$, and define $g: W \rightarrow Y$ to be the unique map. Describe the set $X \sqcup_W Y$. \diamond

Exercise 2.6.2.7. Let $i: R \subseteq X \times X$ be an equivalence relation (see Example 2.1.2.3 for notation). Composing with the projections $\pi_1, \pi_2: X \times X \rightarrow X$, we have two maps $\pi_1 \circ i, \pi_2 \circ i: R \rightarrow X$.

a.) What is the pushout

$$X \xleftarrow{\pi_1 \circ i} R \xrightarrow{\pi_2 \circ i} X?$$

b.) If $i: R \subseteq X \times X$ is not assumed to be an equivalence relation, we can still define the pushout above. Is there a relationship between the pushout $X \xleftarrow{\pi_1 \circ i} R \xrightarrow{\pi_2 \circ i} X$ and the equivalence relation generated by $R \subseteq X \times X$?

\diamond

Lemma 2.6.2.8 (Universal property for pushout). *Suppose given the diagram of sets and functions as below.*

$$\begin{array}{ccc} W & \xrightarrow{u} & Y \\ \downarrow t & & \\ X & & \end{array}$$

For any set A and commutative solid arrow diagram as below (i.e. functions $f: X \rightarrow A$ and $g: Y \rightarrow A$ such that $f \circ t = g \circ u$),

$$\begin{array}{ccccc} & & W & & \\ & & \swarrow t & & \searrow u \\ X & & & & Y \\ & \searrow f & & \swarrow g & \\ & & A & & \\ & \swarrow i_1 & \uparrow \exists! & \searrow i_2 & \\ & & X \sqcup_W Y & & \end{array} \tag{2.39}$$

there exists a unique arrow $\begin{Bmatrix} f \\ g \end{Bmatrix}: X \sqcup_W Y \rightarrow A$ making everything commute,

$$f = \begin{Bmatrix} f \\ g \end{Bmatrix} \circ i_1 \quad \text{and} \quad g = \begin{Bmatrix} f \\ g \end{Bmatrix} \circ i_2.$$

2.6.3 Other finite colimits

Definition 2.6.3.1. [Coequalizer]

Suppose given two parallel arrows

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y. \quad X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y \xrightarrow{q} \text{Coeq}(f, g) \quad (2.40)$$

The *coequalizer of f and g* is the commutative diagram as to the right in (2.40), where we define

$$\text{Coeq}(f, g) := Y / f(x) \sim g(x)$$

i.e. the coequalizer of f and g is the quotient of Y by the equivalence relation generated by $\{(f(x), g(x)) \mid x \in X\} \subseteq Y \times Y$

Exercise 2.6.3.2. Let $X = \mathbb{R}$ be the set of real numbers. What is the coequalizer of the two maps $X \rightarrow X$ given by $x \mapsto x$ and $x \mapsto (x + 1)$ respectively? \diamond

Exercise 2.6.3.3. Find a universal property enjoyed by the coequalizer of two arrows. \diamond

Exercise 2.6.3.4 (Initial object). An initial set is a set S such that for every set A , there exists a unique function $S \rightarrow A$.

- a.) Find an initial set.
- b.) Do you think that the notion *initial set* belongs in this section (Section 2.6)? How so? If coproducts, pushouts, and coequalizers are all colimits, what do colimits have in common?

\diamond

2.7 Other notions in Set

In this section we discuss some left-over notions in the category of Sets.

2.7.1 Retractions

Definition 2.7.1.1. Suppose we have a function $f: X \rightarrow Y$ and a function $g: Y \rightarrow X$ such that $g \circ f = \text{id}_X$. In this case we call f a *retract section* and we call g a *retract projection*.

Exercise 2.7.1.2. Create an olog that includes sets X and Y , and functions $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f = \text{id}_X$ but such that $f \circ g \neq \text{id}_Y$; that is, such that f is a retract section but not an isomorphism. \diamond

2.7.2 Currying

Currying is the idea that when a function takes many inputs, we can input them one at a time or all at once. For example, consider the function that takes a material M and an extension E and returns the force transmitted through the material when it is pulled to that extension. This is a function $e: \text{「a material」} \times \text{「an extension」} \rightarrow \text{「a force」}$. This function takes two inputs at once, but it is convenient to “curry” the second input. Recall

that $\text{Hom}_{\mathbf{Set}}(\ulcorner \text{an extension} \urcorner, \ulcorner \text{a force} \urcorner)$ is the set of theoretical force-extension curves. Currying transforms e into a function

$$e' : \ulcorner \text{a material} \urcorner \rightarrow \text{Hom}_{\mathbf{Set}}(\ulcorner \text{an extension} \urcorner, \ulcorner \text{a force} \urcorner).$$

This is a more convenient way to package the same information.

In fact, it may be convenient to repackage this information another way. For any extension, we may want the function that takes a material and returns how much force it can transmit at that extension. This is a function

$$e'' : \ulcorner \text{an extension} \urcorner \rightarrow \text{Hom}_{\mathbf{Set}}(\ulcorner \text{a material} \urcorner, \ulcorner \text{a force} \urcorner).$$

Notation 2.7.2.1. Let A and B be sets. We sometimes denote the set of functions from A to B by

$$B^A := \text{Hom}_{\mathbf{Set}}(A, B). \quad (2.41)$$

Exercise 2.7.2.2. For a finite set A , let $|A| \in \mathbb{N}$ denote the cardinality of (number of elements in) A . If A and B are both finite (including the possibility that one or both are empty), is it always true that $|B^A| = |B|^{|A|}$? \diamond

Proposition 2.7.2.3 (Currying). *Let A denote a set. For any sets X, Y there is a bijection*

$$\phi : \text{Hom}_{\mathbf{Set}}(X \times A, Y) \xrightarrow{\cong} \text{Hom}_{\mathbf{Set}}(X, Y^A). \quad (2.42)$$

Proof. Suppose given $f : X \times A \rightarrow Y$. Define $\phi(f) : X \rightarrow Y^A$ as follows: for any $x \in X$ let $\phi(f)(x) : A \rightarrow Y$ be defined as follows: for any $a \in A$, let $\phi(f)(x)(a) := f(x, a)$.

We now construct the inverse, $\psi : \text{Hom}_{\mathbf{Set}}(X, Y^A) \rightarrow \text{Hom}_{\mathbf{Set}}(X \times A, Y)$. Suppose given $g : X \rightarrow Y^A$. Define $\psi(g) : X \times A \rightarrow Y$ as follows: for any pair $(x, a) \in X \times A$ let $\psi(g)(x, a) := g(x)(a)$.

Then for any $f \in \text{Hom}_{\mathbf{Set}}(X \times A, Y)$ we have $\psi \circ \phi(f)(x, a) = \phi(f)(x)(a) = f(x, a)$, and for any $g \in \text{Hom}_{\mathbf{Set}}(X, Y^A)$ we have $\phi \circ \psi(g)(x)(a) = \psi(g)(x, a) = g(x)(a)$. Thus we see that ϕ is an isomorphism as desired. \square

Exercise 2.7.2.4. Let $X = \{1, 2\}$, $A = \{a, b\}$, and $Y = \{x, y\}$.

- Write down three distinct elements of $L := \text{Hom}_{\mathbf{Set}}(X \times A, Y)$.
- Write down all the elements of $M := \text{Hom}_{\mathbf{Set}}(A, Y)$.
- For each of the three elements $\ell \in L$ you chose in part (a), write down the corresponding function $\phi(\ell) : X \rightarrow M$ guaranteed by Proposition 2.7.2.3. \diamond

Exercise 2.7.2.5. Let A and B be sets. We know that $\text{Hom}_{\mathbf{Set}}(A, B) = B^A$, so we have a function $\text{id}_{B^A} : \text{Hom}_{\mathbf{Set}}(A, B) \rightarrow B^A$. Look at Proposition 2.7.2.3, making the substitutions $X = \text{Hom}_{\mathbf{Set}}(A, B)$, $Y = B$, and $A = A$. Consider the function

$$\phi^{-1} : \text{Hom}_{\mathbf{Set}}(\text{Hom}_{\mathbf{Set}}(A, B), B^A) \rightarrow \text{Hom}_{\mathbf{Set}}(\text{Hom}_{\mathbf{Set}}(A, B) \times A, B)$$

obtained as the inverse of (2.42). We have a canonical element id_{B^A} in the domain of ϕ^{-1} . We can apply the function ϕ^{-1} and obtain an element $ev = \phi^{-1}(\text{id}_{B^A}) \in \text{Hom}_{\mathbf{Set}}(\text{Hom}_{\mathbf{Set}}(A, B) \times A, B)$, which is itself a function,

$$ev : \text{Hom}_{\mathbf{Set}}(A, B) \times A \rightarrow B.$$

- a.) Describe the function ev in terms of how it operates on elements in its domain.
 b.) Why might one be tempted to denote this function by ev ?

◇

If $n \in \mathbb{N}$ is a natural number, recall from (2.6) that there is a nice set $\underline{n} = \{1, 2, \dots, n\}$. If A is a set, we often make the abbreviation

$$A^n := A^{\underline{n}}. \quad (2.43)$$

Exercise 2.7.2.6. In Example 2.4.1.7 we said that \mathbb{R}^2 is an abbreviation for $\mathbb{R} \times \mathbb{R}$, but in (2.43) we say that \mathbb{R}^2 is an abbreviation for $\mathbb{R}^{\underline{2}}$. Use Exercise 2.1.2.14, Proposition 2.7.2.3, Exercise 2.4.2.10, and the fact that $1+1=2$, to prove that these are isomorphic, $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$.

(The answer to Exercise 2.1.2.14 was $A = \{\odot\}$: i.e. $\text{Hom}_{\text{Set}}(\{\odot\}, X) \cong X$ for all X .)

◇

2.7.3 Arithmetic of sets

Proposition 2.7.3.1 summarizes the properties of products, coproducts, and exponentials, and shows them all in a familiar light, namely that of arithmetic. In fact, one can think of the natural numbers as literally being the isomorphism classes of finite sets—that’s what they are used for in counting. Consider the standard procedure for counting the elements of a set S , say cows in a field: one points to an element in S and simultaneously says “1”, points to another element in S and simultaneously says “2”, and so on until finished. This procedure amounts to nothing more than creating an isomorphism (one-to-one mapping) between S and some set \underline{n} .

Again, the natural numbers are the isomorphism classes of finite sets. Their behavior, i.e. the arithmetic of natural numbers, reflects the behavior of sets. For example the fact that multiplication distributes over addition is a fact about grids of dots as in Example 2.4.1.2. The following proposition lays out such arithmetic properties of sets.

In this proposition, we denote the coproduct of two sets A and B by the notation $A + B$ rather than $A \sqcup B$. It is a reasonable notation in general, and one that is often used.

Proposition 2.7.3.1. *The following isomorphisms exist for any sets A, B , and C (except for one caveat, see Exercise 2.7.3.2).*

- $A + \underline{0} \cong A$
- $A + B \cong B + A$
- $(A + B) + C \cong A + (B + C)$
- $A \times \underline{0} \cong \underline{0}$
- $A \times \underline{1} \cong A$
- $A \times B \cong B \times A$
- $(A \times B) \times C \cong A \times (B \times C)$
- $A \times (B + C) \cong (A \times B) + (A \times C)$

- $A^{\underline{0}} \cong \underline{1}$
- $A^{\underline{1}} \cong A$
- $\underline{0}^A \cong \underline{0}$
- $\underline{1}^A \cong \underline{1}$
- $A^{B+C} \cong A^B \times A^C$
- $(A^B)^C \cong A^{B \times C}$

Exercise 2.7.3.2. Everything in Proposition 2.7.3.1 is true except in one case, namely that of

$$\underline{0}^{\underline{0}}.$$

In this case, we get conflicting answers, because for any set A , including $A = \emptyset = \underline{0}$, we have claimed both that $A^{\underline{0}} \cong \underline{1}$ and that $\underline{0}^A \cong \underline{0}$.

What is the correct answer for $\underline{0}^{\underline{0}}$, based on the definitions of $\underline{0}$ and $\underline{1}$, given in (2.6), and of A^B , given in (2.41)? \diamond

Exercise 2.7.3.3. It is also true of natural numbers that if $a, b \in \mathbb{N}$ and $ab = 0$ then either $a = 0$ or $b = 0$. Is the analogous statement true of all sets? \diamond

Proposition 2.7.3.1 is in some sense about isomorphisms. It says that understanding isomorphisms of sets reduces to understanding natural numbers. But note that there is much more going on in **Set** than isomorphisms; in particular there are functions that are not invertible.

In grade school you probably never saw anything that looked like this:

$$5^3 \times 3 \longrightarrow 5$$

And yet in Exercise 2.7.2.5 we found a function $ev: B^A \times A \rightarrow B$ that exists for any sets A, B . This function ev is not an isomorphism so it somehow does not show up as an equation of natural numbers. But it still has important meaning.¹⁶ In terms of mere number, it looks like we are being told of an important function $\underline{575} \rightarrow \underline{5}$, which is bizarre. The issue here is precisely the one you confronted in Exercise 2.1.2.13.

Exercise 2.7.3.4. Explain why there is a canonical function $\underline{5}^{\underline{3}} \times \underline{3} \longrightarrow \underline{5}$ but not a canonical function $\underline{575} \rightarrow \underline{5}$. \diamond

Slogan 2.7.3.5.

“ It is true that a set is isomorphic to any other set with the same number of elements, but don’t be fooled into thinking that the study of sets reduces to the study of numbers. Functions that are not isomorphisms cannot be captured within the framework of numbers. ”

¹⁶Roughly, the existence of $ev: \underline{5}^{\underline{3}} \times \underline{3} \longrightarrow \underline{5}$ says that given a dot in a $5 \times 5 \times 5$ grid of dots, and given one of the three axes, you can tell me the coordinate of that dot along that axis.

2.7.4 Subobjects and characteristic functions

Definition 2.7.4.1. For any set B , define the *power set of B* , denoted $\mathbb{P}(B)$, to be the set of subsets of B .

Exercise 2.7.4.2.

- How many elements does $\mathbb{P}(\emptyset)$ have?
- How many elements does $\mathbb{P}(\{\odot\})$ have?
- How many elements does $\mathbb{P}(\{1, 2, 3, 4, 5, 6\})$ have?
- Any idea why they may have named it “power set”?

◇

2.7.4.3 Simplicial complexes

Definition 2.7.4.4. Let V be a set and let $\mathbb{P}(V)$ be its powerset. A subset $X \subseteq \mathbb{P}(V)$ is called *downward-closed* if, for every $u \in X$ and every $u' \subseteq u$, we have $u' \in X$. We say that X *contains all atoms* if for every $v \in V$ the singleton set $\{v\}$ is an element of X .

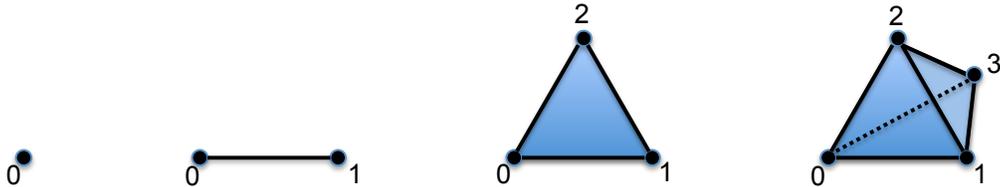
A *simplicial complex* is a pair (V, X) where V is a set and $X \subseteq \mathbb{P}(V)$ is a downward-closed subset that contains all atoms. The elements of X are called *simplices* (singular: *simplex*). Any subset $u \subseteq V$ has a cardinality $|u|$, so we have a function $X \rightarrow \mathbb{N}$ sending each simplex to its cardinality. The set of simplices with cardinality $n + 1$ is denoted X_n and each element $x \in X_n$ is called an *n -simplex*.¹⁷ Since X contains all atoms (subsets of cardinality 1), we have $X_0 \cong V$, and we may also call the 0-simplices *vertices*. We sometimes call the 1-simplices *edges*.¹⁸

Since $X_0 \cong V$, we may denote a simplicial complex (V, X) simply by X .

Example 2.7.4.5. Let $n \in \mathbb{N}$ be a natural number and let $V = \underline{n + 1}$. Define *the n -simplex*, denoted Δ^n , to be the simplicial complex $\mathbb{P}(V) \subseteq \mathbb{P}(V)$, i.e. the whole power set, which indeed is downward-closed and contains all atoms.

We can draw a simplicial complex X by first putting all the vertices on the page as dots. Then for every $x \in X_1$, we see that $x = \{v, v'\}$ consists of 2 vertices, so we draw an edge connecting v and v' . For every $y \in X_2$ we see that $y = \{w, w', w''\}$ consists of 3 vertices, so we draw a (filled-in) triangle connecting them. All three edges will be drawn too because X is assumed to be downward closed.

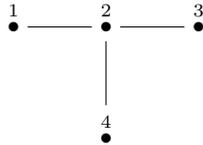
Thus, the 0-simplex Δ^0 , the 1-simplex Δ^1 , the 2-simplex Δ^2 , and the 3-simplex Δ^3 are drawn here:



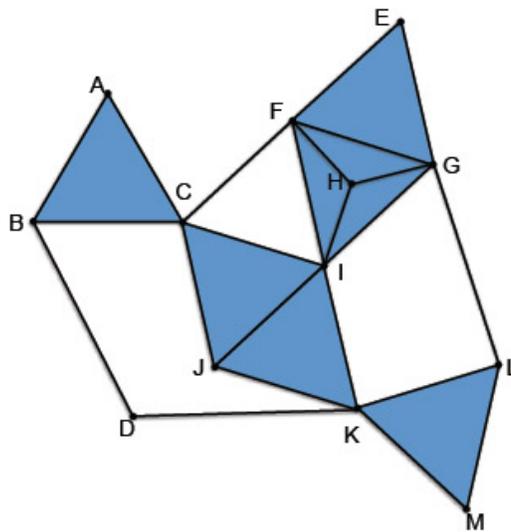
¹⁷It is annoying at first that the set of subsets with cardinality 1 is denoted X_0 , etc. But this is standard convention because as we will see, X_n will be n -dimensional.

¹⁸The reason we wrote $X_0 \cong V$ rather than $X_0 = V$ is that X_0 is the set of 1-element subsets of V . So if $V = \{a, b, c\}$ then $X_0 = \{\{a\}, \{b\}, \{c\}\}$. This is really just pedantry.

The n -simplices for various n 's are in no way all of the simplicial complexes. In general a simplicial complex is a union or “gluing together” of simplices in a prescribed manner. For example, consider the simplicial complex X with vertices $X_0 = \{1, 2, 3, 4\}$, edges $X_1 = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$, and no higher simplices $X_2 = X_3 = \dots = \emptyset$. We might draw X as follows:



Exercise 2.7.4.6. Let X be the following simplicial complex, so that $X_0 = \{A, B, \dots, M\}$.



In this case X_1 consists of elements like $\{A, B\}$ and $\{D, K\}$ but not $\{D, J\}$.

Write out X_2 and X_3 (hint: the drawing of X indicates that X_3 should have one element). ◇

Exercise 2.7.4.7. The 2-simplex Δ^2 is drawn as a filled-in triangle with vertices $V = \{1, 2, 3\}$. There is a simplicial complex $X = \partial\Delta^2$ that would be drawn as an empty triangle with the same set of vertices.

- a.) Draw Δ^2 and X side by side and make clear the difference.
- b.) Write down the data for X as a simplicial complex. In other words what are the sets $X_0, X_1, X_2, X_3, \dots$?

◇

2.7.4.8 Subobject classifier

Definition 2.7.4.9. Define the *subobject classifier* for **Set**, denoted Ω , to be the set $\Omega := \{True, False\}$, together with the function $\{\odot\} \rightarrow \Omega$ sending the unique element to *True*.

Proposition 2.7.4.10. *Let B be a set. There is an isomorphism*

$$\phi: \text{Hom}_{\mathbf{Set}}(B, \Omega) \xrightarrow{\cong} \mathbb{P}(B).$$

Proof. Given a function $f: B \rightarrow \Omega$, let $\phi(f) = \{b \in B \mid f(b) = \text{True}\} \subseteq B$. We now construct a function $\psi: \mathbb{P}(B) \rightarrow \text{Hom}_{\mathbf{Set}}(B, \Omega)$ to serve as the inverse of ϕ . Given a subset $B' \subseteq B$, define $\psi(B'): B \rightarrow \Omega$ as follows:

$$\psi(i)(b) = \begin{cases} \text{True} & \text{if } b \in B', \\ \text{False} & \text{if } b \notin B'. \end{cases}$$

One checks easily that ϕ and ψ are mutually inverse. □

Definition 2.7.4.11 (Characteristic function). Given a subset $B' \subseteq B$, we call the corresponding function $B \rightarrow \Omega$ the *characteristic function of B' in B* .

Let B be any set and let $\mathbb{P}(B)$ be its power set. By Proposition 2.7.4.10 there is a bijection between $\mathbb{P}(B)$ and Ω^B . Since Ω has cardinality 2, the cardinality of $\mathbb{P}(B)$ is $2^{|B|}$, which explains the correct answer to Exercise 2.7.4.2.

Exercise 2.7.4.12. Let $f: A \rightarrow \Omega$ denote the characteristic function of some $A' \subseteq A$, and define $A'' \subseteq A$ to be its complement, $A'' := A - A'$ (i.e. $a \in A''$ if and only if $a \notin A'$).

- a.) What is the characteristic function of $A'' \subseteq A$?
- b.) Can you phrase it in terms of some function $\Omega \rightarrow \Omega$?

◇

2.7.5 Surjections, injections

The classical definition of injections and surjections involves elements, which we give now. But a more robust notion involves all maps and will be given in Proposition 2.7.5.4.

Definition 2.7.5.1. Let $f: X \rightarrow Y$ be a function. We say that f is *surjective* if, for all $y \in Y$ there exists some $x \in X$ such that $f(x) = y$. We say that f is *injective* if, for all $x \in X$ and all $x' \in X$ with $f(x) = f(x')$ we have $x = x'$.

A function that is both injective and surjective is called *bijective*.

Remark 2.7.5.2. It turns out that a function that is bijective is always an isomorphism and that all isomorphisms are bijective. We will not show that here, but it is not too hard; see for example [Big, Theorem 5.4].

Definition 2.7.5.3 (Monomorphisms, epimorphisms). Let $f: X \rightarrow Y$ be a function.

We say that f is a *monomorphism* if for all sets A and pairs of functions $g, g': A \rightarrow X$,

$$\begin{array}{ccc} & g & \\ & \curvearrowright & \\ A & & X \xrightarrow{f} Y \\ & \curvearrowleft & \\ & g' & \end{array}$$

if $f \circ g = f \circ g'$ then $g = g'$.

We say that f is an *epimorphism* if for all sets B and pairs of functions $h, h': Y \rightarrow B$,

$$X \xrightarrow{f} Y \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{h'} \end{array} B$$

if $h \circ f = h' \circ f$ then $h = h'$.

Proposition 2.7.5.4. *Let $f: X \rightarrow Y$ be a function. Then f is injective if and only if it is a monomorphism; f is surjective if and only if it is an epimorphism.*

Proof. If f is a monomorphism it is clearly injective by putting $A = \{\odot\}$. Suppose that f is injective and let $g, g': A \rightarrow X$ be functions such that $f \circ g = f \circ g'$, but suppose for contradiction that $g \neq g'$. Then there is some element $a \in A$ such $g(a) \neq g'(a) \in X$. But by injectivity $f(g(a)) \neq f(g'(a))$, contradicting $f \circ g = f \circ g'$.

Suppose that $f: X \rightarrow Y$ is an epimorphism and choose some $y_0 \in Y$ (noting that if Y is empty then the claim is vacuously true). Let $h: Y \rightarrow \Omega$ denote the characteristic function of the subset $\{y_0\} \subseteq Y$ and let $h': Y \rightarrow \Omega$ denote the characteristic function of $\emptyset \subseteq Y$; note that $h(y) = h'(y)$ for all $y \neq y_0$. Then since f is an epimorphism and $h \neq h'$, we must have $h \circ f \neq h' \circ f$, so there exists $x \in X$ with $h(f(x)) \neq h'(f(x))$, which implies that $f(x) = y_0$. This proves that f is surjective.

Finally, suppose that f is surjective, and let $h, h': Y \rightarrow B$ be functions with $h \circ f = h' \circ f$. For any $y \in Y$, there exists some $x \in X$ with $f(x) = y$, so $h(y) = h(f(x)) = h'(f(x)) = h'(y)$. This proves that f is an epimorphism. □

Proposition 2.7.5.5. *Let $f: X \rightarrow Y$ be a monomorphism. Then for any function $g: A \rightarrow Y$, the top map $f': X \times_Y A \rightarrow A$ in the diagram*

$$\begin{array}{ccc} X \times_Y A & \xrightarrow{f'} & A \\ g' \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

is a monomorphism.

Proof. To show that f' is a monomorphism, we take an arbitrary set B and two maps $m, n: B \rightarrow X \times_Y A$ such that $f' \circ m = f' \circ n$, denote that function by $p := f' \circ m: B \rightarrow A$. Now let $q = g' \circ m$ and $r = g' \circ n$. The diagram looks like this:

$$\begin{array}{ccccc} & & & & p \\ & & & & \curvearrowright \\ B & \xrightarrow{m} & X \times_Y A & \xrightarrow{f'} & A \\ & \searrow n & & \lrcorner & \downarrow g \\ & & & & \\ & \searrow q & & & \\ & & X & \xrightarrow{f} & Y \\ & \searrow r & & & \end{array}$$

We have that

$$f \circ q = f \circ g' \circ m = g \circ f' \circ m = g \circ f' \circ n = f \circ g' \circ n = f \circ r$$

But we assumed that f is a monomorphism so this implies that $q = r$. By the universal property of pullbacks, Lemma 2.5.1.14, we have $m = n$. \square

Exercise 2.7.5.6. Show, in analogy to Proposition 2.7.5.5, that pushouts preserve epimorphisms. \diamond

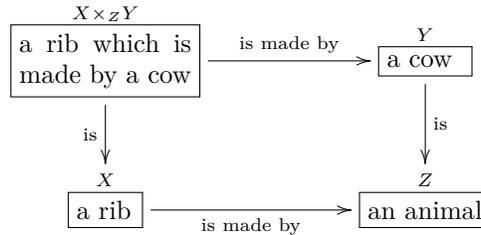
Example 2.7.5.7. Suppose an olog has a fiber product square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{g'} & Y \\ f' \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Z \end{array}$$

such that f is intended to be an injection and g is any map.¹⁹ In this case, there are nice labeling systems for f' , g' , and $X \times_Z Y$. Namely:

- “is” is an appropriate label for f' ,
- the label for g is an appropriate label for g' ,
- (the label for X , then “which”, then the label for g , then the label for Y) is an appropriate label for $X \times_Z Y$.

To give an explicit example,



Corollary 2.7.5.8. *Let $i: A \rightarrow X$ be a monomorphism. Then there is a fiber product square of the form*

$$\begin{array}{ccc} A & \xrightarrow{f'} & \{\odot\} \\ i \downarrow & \lrcorner & \downarrow \text{True} \\ X & \xrightarrow{f} & \Omega. \end{array} \quad (2.44)$$

Proof. Let $X' \subseteq X$ denote the image of i and let $f: X \rightarrow \Omega$ denote the characteristic function of $X' \subseteq X$. Then it is easy to check that Diagram 2.44 is a pullback. \square

Exercise 2.7.5.9. Consider the subobject classifier Ω , the singleton $\{\odot\}$ and the map $\{\odot\} \xrightarrow{\text{True}} \Omega$ from Definition 2.7.4.9. Look at diagram 2.44 and in the spirit of Exercise 2.7.5.7, come up with a label for Ω , a label for $\{\odot\}$, and a label for True . Given a label for X and a label for f , come up with a label for A , a label for i and a label for f' , such that the English smoothly fits the mathematics. \diamond

¹⁹Of course, this diagram is symmetrical, so the same ideas hold if g is an injection and f is any map.

2.7.6 Multisets, relative sets, and set-indexed sets

In this section we prepare ourselves for considering categories other than **Set**, by looking at some categories related to **Set**.

2.7.6.1 Multisets

Consider the set X of words in a given document. If $WC(X)$ is the wordcount of the document, we will not generally have $WC(X) = |X|$. The reason is that a set cannot contain the same element more than once, so words like “the” might be undercounted in $|X|$. A *multiset* is a set in which elements can be assigned a multiplicity, i.e. a number of times they are to be counted.

But if X and Y are multisets, what is the appropriate type of mapping from X to Y ? Since every set is a multiset (in which each element has multiplicity 1), let’s restrict ourselves to notions of mapping that agree with the usual one on sets. That is, if multisets X and Y happen to be sets then our mappings $X \rightarrow Y$ should just be functions.

Exercise 2.7.6.2.

- a.) Come up with some notion of mapping for multisets that generalizes functions when the notion is restricted to sets.
- b.) Suppose that $X = (1, 1, 2, 3)$ and $Y = (a, b, b, b)$, i.e. $X = \{1, 2, 3\}$ with 1 having multiplicity 2, and $Y = \{a, b\}$ with b having multiplicity 3. What are all the maps $X \rightarrow Y$ in your notion?

◇

In Chapter 4 we will be getting to the definition of category, and you can test whether your notion of mapping in fact defines a category. Here is my definition of mapping for multisets.

Definition 2.7.6.3. A *multiset* is a sequence $X := (E, B, \pi)$ where E and B are sets and $\pi: E \rightarrow B$ is a surjective function. We refer to E as the set of *element instances* of X , we refer to B as the set of *element names* of X , and we refer to π as the *naming function* for X . Given an element name $x \in B$, let $\pi^{-1}(x) \subseteq E$ be the preimage; the number of elements in $\pi^{-1}(x)$ is called the *multiplicity* of x .

Suppose that $X = (E, B, \pi)$ and $X' = (E', B', \pi')$ are multisets. A *mapping from X to Y* , denoted $f: X \rightarrow Y$, consists of a pair (f_1, f_0) such that $f_1: E \rightarrow E'$ and $f_0: B \rightarrow B'$ are functions and such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f_1} & E' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f_0} & B'. \end{array} \quad (2.45)$$

Exercise 2.7.6.4. Suppose that a pseudo-multiset is defined to be almost the same as a multiset, except that π is not required to be surjective.

- a.) Write down a pseudo-multiset that is not a multi-set.
- b.) Describe the difference between the two notions in terms of multiplicities.

- c.) Complexity of names aside, which do you think is a more useful notion: multiset or pseudo-multisets?

◇

Exercise 2.7.6.5. Consider the multisets described in Exercise 2.7.6.2.

- a.) Write each of them in the form (E, B, π) , as in Definition 2.7.6.3.
 b.) In terms of the same definition, what are the mappings $X \rightarrow Y$?
 c.) If we remove the restriction that diagram 2.45 must commute, how many mappings $X \rightarrow Y$ are there?

◇

2.7.6.6 Relative sets

Let's continue with our ideas from multisets, but now suppose that we have a fixed set B of names that we want to keep once and for all. Whenever someone discusses a set, each element must have a name in B . And whenever someone discusses a mapping, it must preserve the names. For example, if B is the set of English words, then every document consists of an ordered set mapping to B (e.g. $1 \mapsto \text{Suppose}, 2 \mapsto \text{that}, 3 \mapsto \text{we}$, etc.) A mapping from document A to document B would send each word found somewhere in A to the same word found somewhere in B . This notion is defined carefully below.

Definition 2.7.6.7 (Relative set). Let B be a set. A *relative set over B* , or simply a *set over B* , is a pair (E, π) such that E is a set and $\pi: E \rightarrow B$ is a function. A *mapping of relative sets over B* , denoted $f: (E, \pi) \rightarrow (E', \pi')$, is a function $f: E \rightarrow E'$ such that the triangle below commutes, i.e. $\pi = \pi' \circ f$,

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow \pi & \swarrow \pi' \\ & & B \end{array}$$

Exercise 2.7.6.8. Given sets X, Y, Z and functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we can compose them to get a function $X \rightarrow Z$. If B is a set, if $(X, p), (Y, q)$, and (Z, r) are relative sets over B , and if $f: (X, p) \rightarrow (Y, q)$ and $g: (Y, q) \rightarrow (Z, r)$ are mappings, is there a reasonable notion of composition such that we get a mapping of relative sets $(X, p) \rightarrow (Z, r)$? Hint: draw diagrams. ◇

Exercise 2.7.6.9.

- a.) Let $\{\odot\}$ denote a set with one element. What is the difference between sets over $\{\odot\}$ and simply sets?
 b.) Describe the sets relative to \emptyset . How many are there?

◇

2.7.6.10 Indexed sets

Let A be a set. Suppose we want to assign to each element $a \in A$ a set S_a . This is called an A -indexed set. In category theory we are always interested in the legal mappings between two different structures of the same sort, so we need a notion of A -indexed mappings; we do the “obvious thing”.

Example 2.7.6.11. Let C be a set of classrooms. For each $c \in C$ let P_c denote the set of people in classroom c , and let S_c denote the set of seats (chairs) in classroom c . Then P and S are C -indexed sets. The appropriate kind of mapping between them respects the indexes. That is, a mapping of multi-sets $P \rightarrow S$ should, for each classroom $c \in C$, be a function $P_c \rightarrow S_c$.²⁰

Definition 2.7.6.12. Let A be a set. An A -indexed set is a collection of sets S_a , one for each element $a \in A$; for now we denote this by $(S_a)_{a \in A}$. If $(S'_a)_{a \in A}$ is another A -indexed set, a *mapping of A -indexed sets from $(S_a)_{a \in A}$ to $(S'_a)_{a \in A}$* , denoted

$$(f_a)_{a \in A}: (S_a)_{a \in A} \rightarrow (S'_a)_{a \in A}$$

is a collection of functions $f_a: S_a \rightarrow S'_a$, one for each element $a \in A$.

Exercise 2.7.6.13. Let $\{\odot\}$ denote a one element set. What are $\{\odot\}$ -indexed sets and mappings between them? ◇

Exercise 2.7.6.14. There is a strong relationship between A -indexed sets and relative sets over A . What is it? ◇

²⁰If we wanted to allow people from any classroom to choose a chair from just any classroom, category theory would tell us to reconsider P and S as sets, forgetting their indices. See Section 5.1.4.7.

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18.S996 Category Theory for Scientist
Spring 2013

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